Chapter 14. Partial Derivatives

14.1. Functions of Several Variables

Note. We now consider functions whose domains are sets of ordered pairs, ordered triples, or in general ordered n-tuples of real numbers, and whose ranges are subsets of the real numbers. For example, a function of two variables might be of the form $F(x, y) = x^2 + y^2$. You have seen this when defining what it means for a function $y = f(x)$ to be *implicit* to an equation $F(x, y) = 0$ (see page 170). We now define this more clearly.

Definition. Suppose D is a set of n-tuples of real numbers $\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$. A *real-valued function* f on D is a rule that assigns a unique (single) real number $w = f(x_1, x_2, \ldots, x_n)$ to each element in D. The set D is the function's *domain*. The set of w-values taken on by f is the function's *range*. The symbol w is the *dependent variable* of f, and f is said to be a function of the n *independent variables* x_1 to x_n.

Note. We can think of f as a mapping from the "space" $\mathbb{R}^n$ of ordered n-tuples of real numbers to the set $\mathbb{R}$ of real numbers.

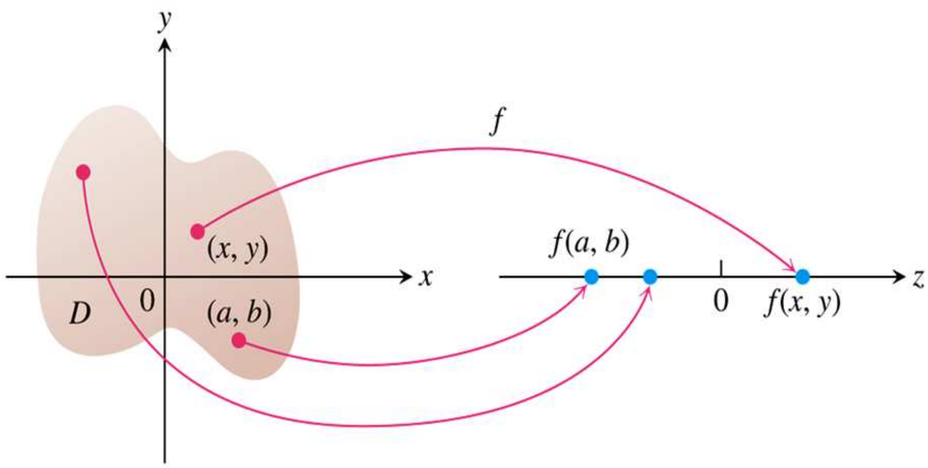
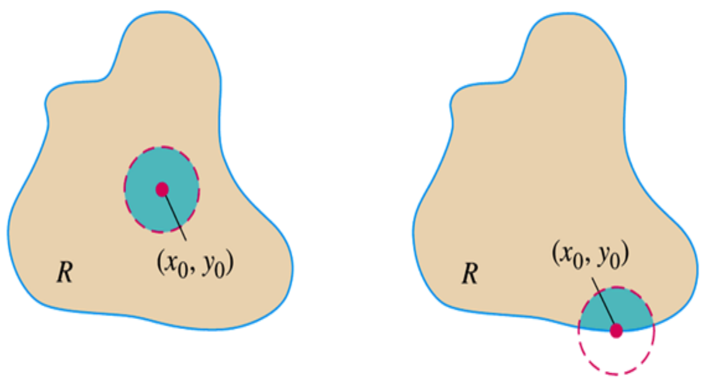


Figure 14.1, page 766

Example. Page 771, number 6.

Definition. A point (x_0, y_0) in a region R in the xy-plane is an *interior point* of R if it is the center of a disk of positive radius that lies entirely in R. A point (x_0, y_0) is a *boundary point* of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. A point (x_0, y_0) is a *limit point* of R if every disk centered at (x_0, y_0) contains a point that lies in R other than (x_0, y_0) itself. (Boundary points and limit points may or may not be in R). The interior points of a region make up the *interior* of the region. The region's boundary point make up its *boundary*. A region is *open* if it consists entirely of interior points. A region is *closed* if it contains all of its boundary points. The *closure* of a region consists of all the points in the set and all limit points of the set. (NOTE: The textbook does not include the definition of *limit point* and *closure*—so this is **bonus education**!)



(a) Interior point (b) Boundary point

Figure 14.2, page 767

Definition. A region in the plane is *bounded* if it lies inside a disk of fixed radius. A region is *unbounded* if it is not bounded.

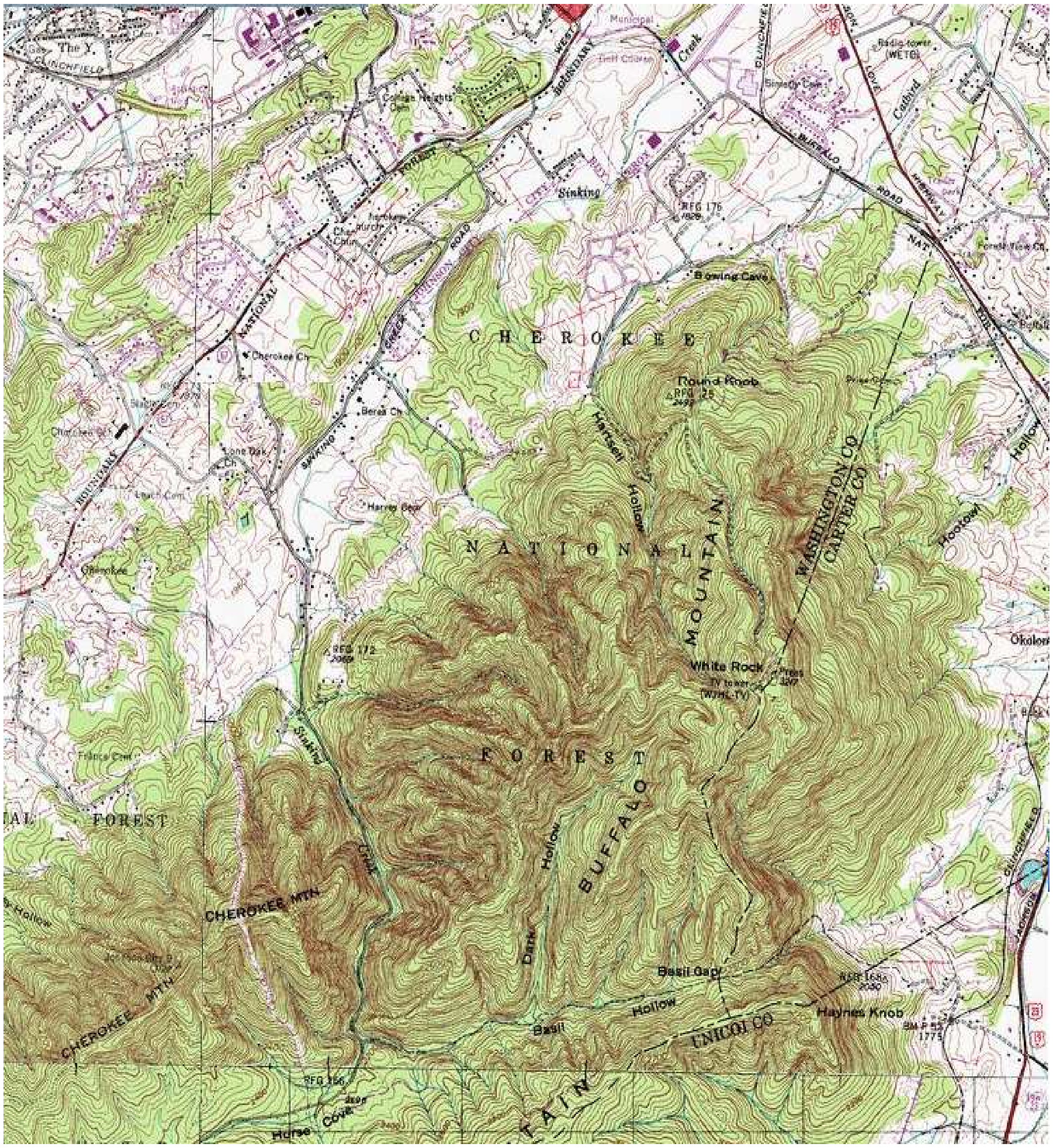
Example. Page 771, number 28.

Note. There are two ways to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value (we concentrate on this technique). The other is to sketch the surface $z = f(x, y)$ in space.

Definition. The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a *level curve* of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the *graph* of f. The graph of f is also called the *surface* $z = f(x, y)$.

Example. Page 772, number 40.

A relief map of Johnson City and Buffalo Mountain (from Google Maps).



A topographic map of Johnson City and Buffalo Mountain (from TopoQuest.com). Notice the distortion on the left resulting from pasting together two different maps. The topographic lines are level curves for the "surface" of the land.

Field Trip! Look out a south-facing window at Buffalo Mountain. This view, combined with the relief map and the topographic map, give three different ways to describe this surface.

Definition. The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a *level surface* of f.

Example. Page 768, Example 4. Describe the level surfaces of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

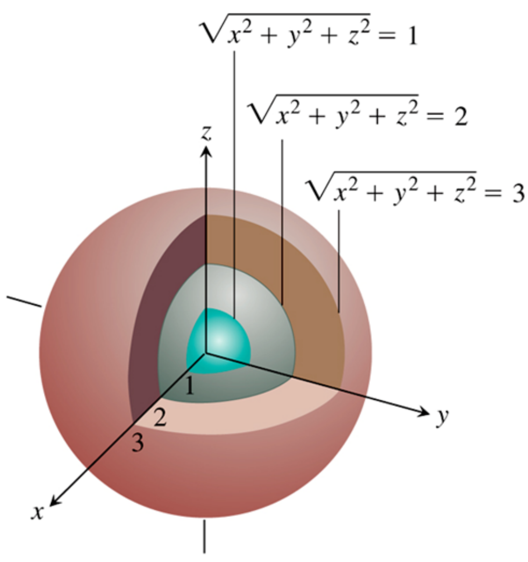


Figure 14.8, page 769

Definition. A point (x_0, y_0, z_0) in a region R in (3-D) space is an *interior point* of R if it is the center of a solid ball (by "solid ball" we mean the set of points lying within a distance $r > 0$ of a given point) that lies entirely in R. A point (x_0, y_0, z_0) is a *boundary point* of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside of R. A point (x_0, y_0, z_0) is a *limit point* of R if every solid ball centered at (x_0, y_0) contains a point that lies in R other than (x_0, y_0, z_0) itself. (Boundary points and limit points may or may not be in R). The *interior* of R is the set of interior points of R. The *boundary* of R is the set of boundary points of R. A region is *open* if it consists entirely of interior points. A region is *closed* if it contains all of its boundary points. The *closure* of a region consists of all the points in the set and all limit points of the set. (NOTE: Again, the textbook does not include the definition of *limit point* and *closure*.)

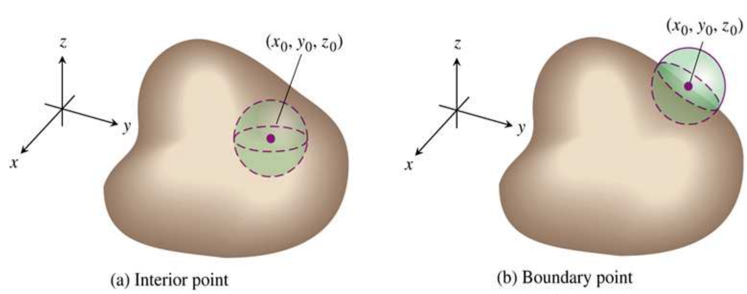


Figure 14.9, page 769

Note. On page 770, the text gives three computer generated surfaces for some "interesting" functions. Software to generate such surfaces is standard these days and these images were likely generated with either *Maple*® or *Mathematica*®.

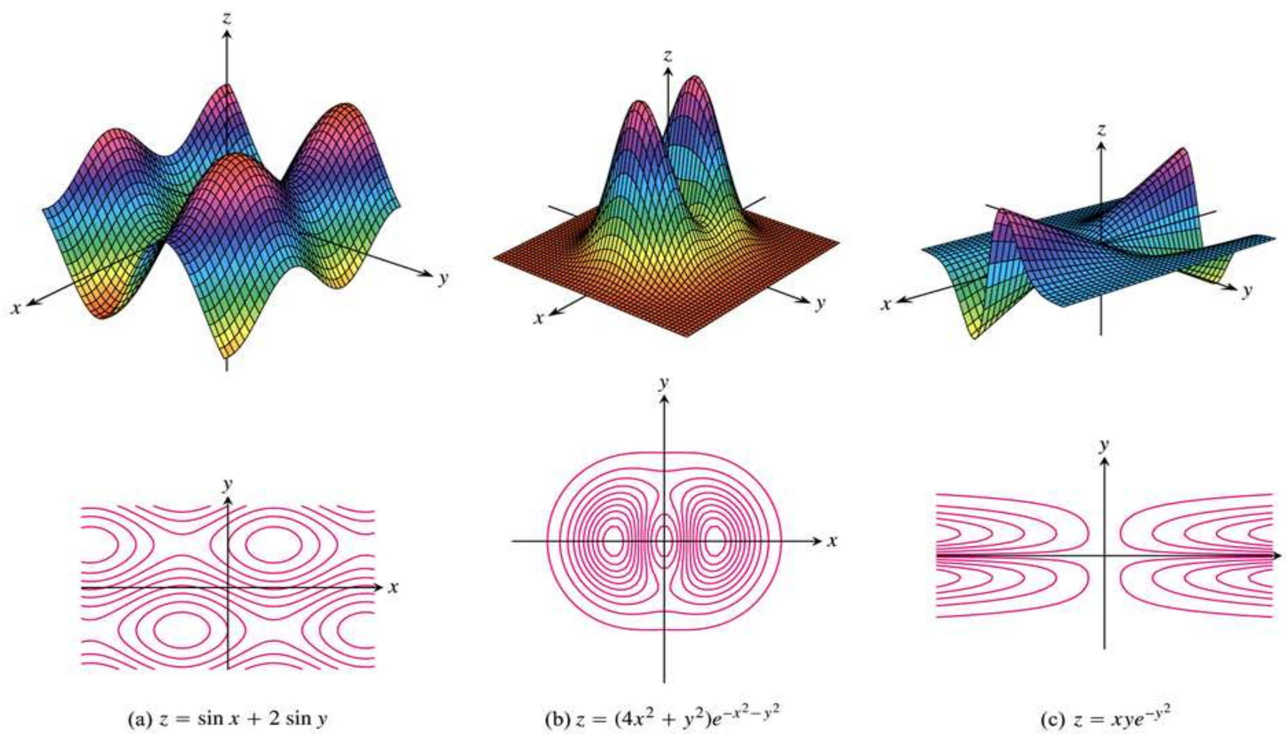


Figure 14.11, page 770