Chapter 14. Partial Derivatives

14.3. Partial Derivatives

Note. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative.

Definition. The *partial derivative of* $f(x, y)$ *with respect to* x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx}[f(x, y_0)]\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

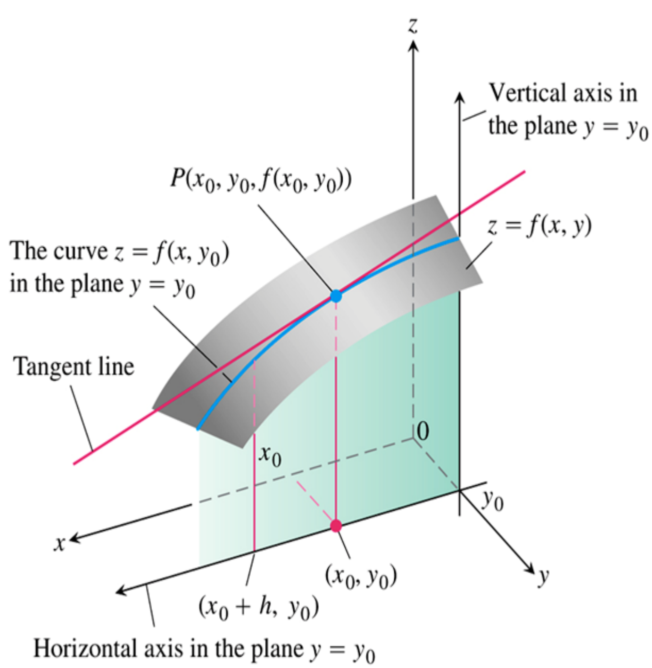

Figure 14.15, page 783

Note. There are several standard notations for the partial derivative of $z = f$ with respect to x:

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \text{ and } f_x, \frac{\partial f}{\partial x}, z_x, \text{ or } \frac{\partial z}{\partial x}.$$

Definition. The *partial derivative of $f(x, y)$ with respect to y* at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy}[f(x_0, y)]\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

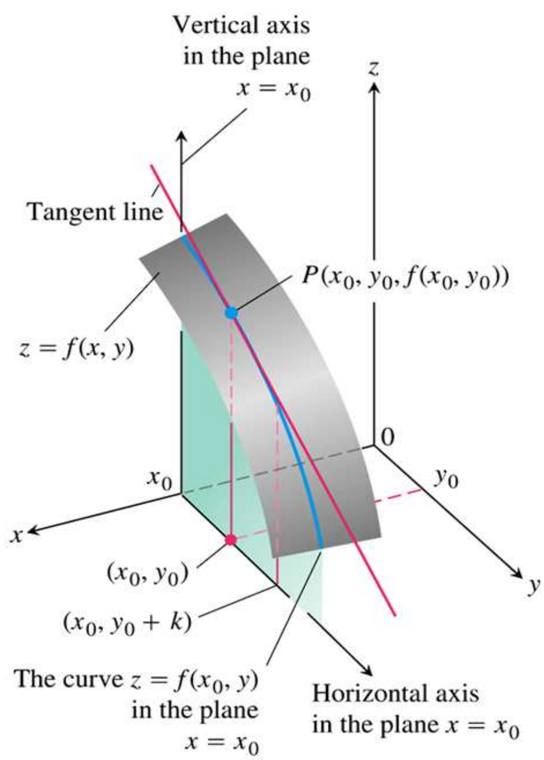

Figure 14.16, page 784

Note. There are several standard notations for the partial derivative of $z = f$ with respect to y:

$$\frac{\partial f}{\partial y}(x_0, y_0) \text{ or } f_y(x_0, y_0), \text{ and } f_y, \frac{\partial f}{\partial y}, z_y, \text{ or } \frac{\partial z}{\partial y}.$$

Note. Notice that we can use the two partial derivatives, $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, to find lines tangent to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$. We will use this later to find the equation of a tangent plane to a surface (in Section 14.6).

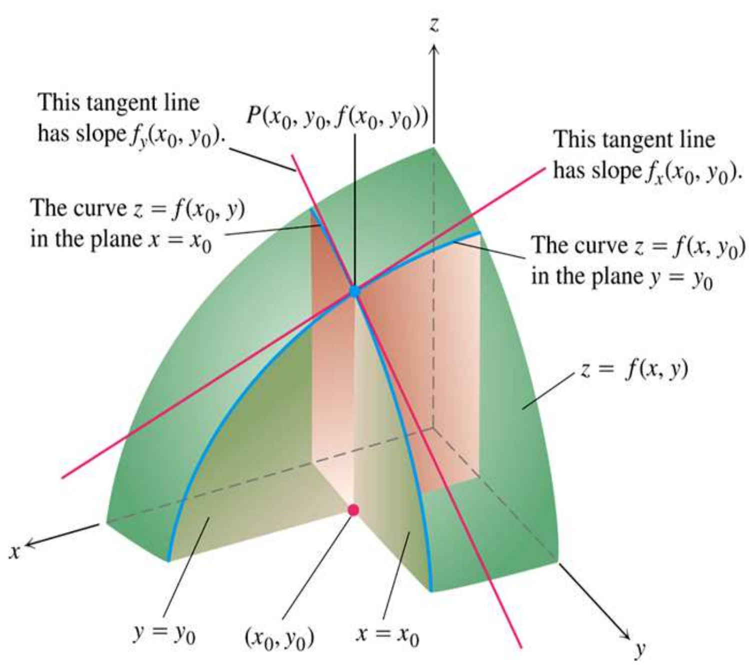

Figure 14.17, page 784

Example. Fortunately, partial derivatives are easy to calculate! Page 790, numbers 2, 12, and 16.

Note. Functions of three variables are partially differentiated in a similar way.

Example. Page 791, number 38.

Note. As with ordinary derivatives, we can take higher order partial derivatives. There are four possible second order derivatives of $f(x, y)$:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \frac{\partial^2 f}{\partial y^2} = f_{yy}, \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \text{ and } \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

Notice the particular order in the last two second order partial derivatives (which are called *mixed* partial derivatives). For example,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{yx} = (f_y)_x.$$

Example. Page 791, number 44.

Note. The mixed partials f_{xy} and f_{yx} may not be equal. However, they often are as given in the following theorem.

Theorem 2. The Mixed Derivative Theorem (Clairaut's Theorem).

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example. Page 791, number 56.

Note. Of course, we can take higher order partial derivatives as well. We just need to (maybe) be careful about the order of differentiation. When using the partial symbol ∂ in the fractional notation, derivatives are calculated by reading the variables from right to left, whereas when we use the subscript notation, the order of differentiation is read from left to right. For example (as the text mentions on page 789):

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \text{ and } \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}.$$

Note. Recall from Calculus 1 that if a function is differentiable at a point, then it is continuous at that point (Theorem 1 on page 131). We want a similar result for functions of several variables. In Example 8 on page 787, it is shown that the function

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

is not continuous at $(0,0)$, yet both of the partial derivatives exist at $(0,0)$. So to get continuity at a point, we need a condition stronger than the existence of the partial derivatives.

Definition. A function $z = f(x,y)$ is *differentiable* at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f *differentiable* if it is differentiable at every point in its domain, and say that its graph $z = f(x,y)$ is a *smooth surface.*

Note. Theorem 3 on page 790 inspires this definition and the role of the ϵ's are given in the proof of Theorem 3 in Appendix 9. We are not interested in the level of detail, but we are interested in the following two results which are much easier to understand than Theorem 3.

Corollary of Theorem 3. If the partial derivatives f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Theorem 4. Differentiability Implies Continuity.
If a function $f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0). (Remember that "differentiable" here means what is stated in the definition above.)

Example. Page 792, number 74.