Chapter 14. Partial Derivatives

14.9. Taylor's Formula for Two Variables

Note. We now justify the Second Derivative Test from section 14.7.

Note. Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$. Let h and k be increments small enough to put the point $S(a+h, b+k)$ and the line segment joining it to P inside R. We parametrize the segment PS as

$$x = a + th, \quad y = b + tk, \quad t \in [0, 1].$$

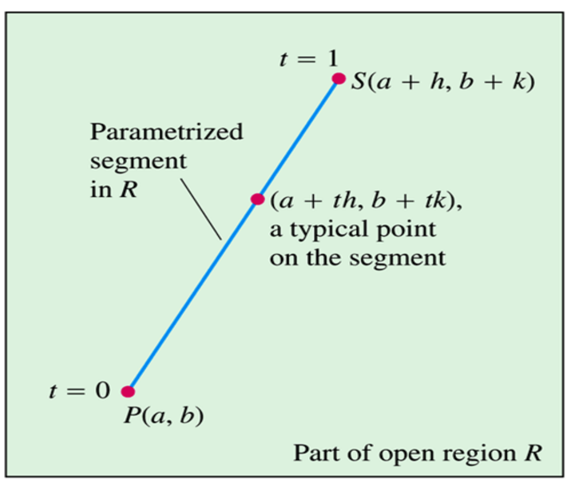

Figure 14.57, Page 838

Define $F(t) = f(a + th, b + tk)$. The Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable (by assumption), F' is a differentiable function of t and

$$F'' = \frac{\partial F'}{\partial x}\frac{dx}{dt} + \frac{\partial F'}{\partial y}\frac{dy}{dt} = \frac{\partial}{\partial x}[hf_x + kf_y] \cdot h + \frac{\partial}{\partial y}[hf_x + kf_y] \cdot k$$
$$= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}.$$

Since F and F' are continuous on $[0, 1]$ and F' is differentiable on $(0, 1)$, we can apply Taylor's Theorem (Theorem 10.23, page 607) with $n = 2$ and $a = 0$ to obtain

$$F(1) = F(0) + F'(0)(1 - 0) + F''(c)\frac{(1-0)^2}{2} = F(0) + F'(0) + \frac{1}{2}F''(0)$$

for some c between 0 and 1. Rewriting in terms of f gives

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + hf_x(a, b) + kf_y(a, b) \\ &\quad + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,b+ck)}. \end{aligned} \qquad (2)$$

Since $f_x(a, b) = f_y(a, b) = 0$, this reduces to

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,b+ck)}.$$

The presence of an extremum of f at (a, b) is determined by the sign of $f(a + h, b + k) - f(a, b)$. From the previous equation, we see that this is the same as the sign of

$$Q(c) \equiv \left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,b+ck)}.$$

If $Q(0) \neq 0$, the sign of $Q(c)$ will be the same as the sign of $Q(0)$ for sufficiently small values of h and k. We can predict the sign of

$$Q(0) = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$$

from the signs of f_{xx} and $f_{xx}f_{yy} - f_{xy}^2$ at (a,b). Multiply both sides of the previous equation by f_{xx} and rearrange the right-hand side to get

$$f_{xx}Q(0) = (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - f_{xy}^2)k^2.$$

We see that

1. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) then $Q(0) < 0$ for all sufficiently small nonzero values of h and k, and f has a *local maximum* value at (a,b).

2. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) then $Q(0) > 0$ for all sufficiently small nonzero values of h and k, and f has a *local minimum* value at (a,b).

3. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) there are combinations of arbitrarily small nonzero values of h and k for which $Q(0) > 0$, and other values for which $Q(0) < 0$. Arbitrarily close to the point $P_0(a, b, f(a, b))$ on the surface $z = f(x, y)$ there are points above P_0 and points below P_0, so f has a *saddle point* at (a, b).

4. If $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b), another test is needed.

Note. This justifies the Second Derivative Test (Theorem 14.11, page 823).

Note. We now justify the error formula for linearizations as given in section 14.6. Assume the function f has continuous second partial derivatives throughout an open set containing a closed rectangular region R centered at (x_0, y_0). Let the number M be an upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R. The inequality we want comes from equation (2) above. We substitute x_0 and y_0 for a and b, and $x - x_0$ and $y - y_0$ for h and k (resp.), and rearrange the result as

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{\text{linearization } L(x,y)}$$

$$\underbrace{+\frac{1}{2}\left((x - x_0)^2 + f_{xx} + 2(x - x_0)(y - y - 0)f_{xy} + (y - y_0)^2 f_{yy}\right)\Big|_{(x_0+c(x-x_0),y_0+c(y-y_0))}}_{\text{error } E(x,y)}.$$

This equation reveals that

$$|E| \leq \frac{1}{2}\left(|x - x_0|^2|f_{xx}| + 2|x - x_0|\,|y - y_0|\,|f_{xy}| + |y - y_0||f_{yy}|\right).$$

Here, if M is an upper bound for the values of $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$ on R,

$$|E| \leq \frac{1}{2}\left(|x-x_0|^2 M + 2|x-x_0\,|y-y_0|M + |y-y_0|^2 M\right)$$

$$= \frac{1}{2}M(|x-x_0| + y-y_0|)^2.$$

Note. This justifies the standard linear approximation of $f(x,y)$ at (x_0, y_0) and the error of this approximation, as given in section 14.6.

Note. We can get a higher degree of approximation of $f(x,y)$ as follows.

Theorem. Taylor's Formula for $f(x,y)$ at the Origin. Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at the point $(0,0)$. then throughout R,

$$f(x,y) = f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_x x + 2xyf_{xy} + y^2 f_{yy})$$

$$+\frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y\frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right)$$

$$+\frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y\frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,cy)}.$$

Example. Page 842, number 2.