Chapter 15. Multiple Integrals

15.4. Double Integrals in Polar Form

Note. Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fanshaped region Q defined by $\{(r, \theta) \mid r \in [0, a], \theta \in [\alpha, \beta]\}$.

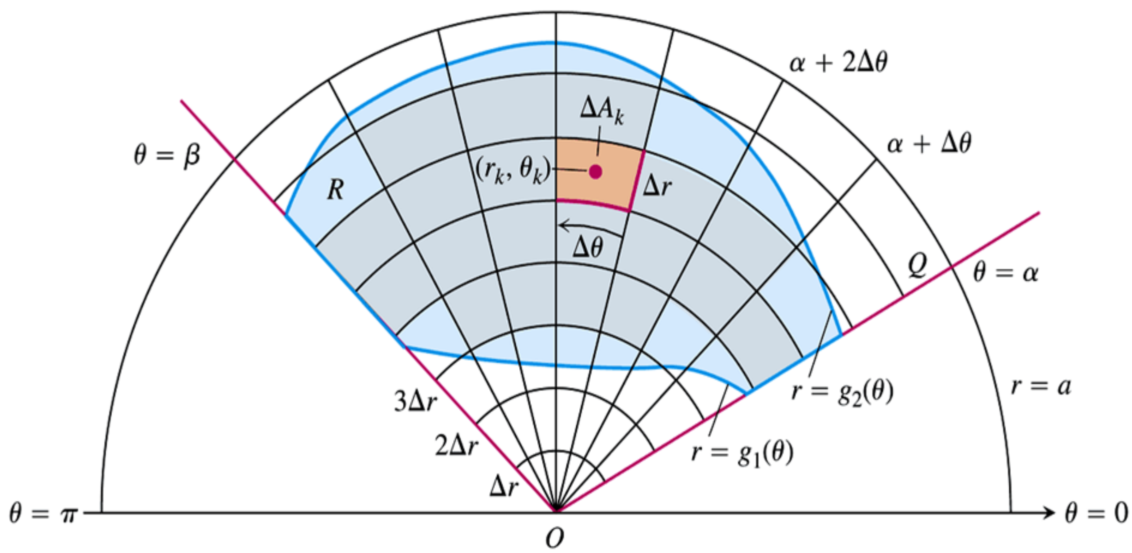

Figure 15.21, Page 871

Note. We cover Q by a grid of a circular arcs and rays. The arcs are cut from circles centered at the origin, with radii Δr, $2\Delta r$, $\ldots$, $m\Delta r$, where $\Delta r = a/m$. The rays are given by:

$$\theta = \alpha, \theta = \alpha + \Delta\theta, \theta = \alpha + 2\Delta\theta, \ldots, \theta = \alpha + m'\Delta\theta = \beta$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles." We number the polar rectangles that lie inside R, calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let (r_k, θ_k) be any pont in the polar rectangle whose area is ΔA_k. We then form the sum $S_n = \displaystyle\sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k$. If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ for to zero. The limit is called the double integral of f over R. We define the *norm* $\|P\|$ of this partition of the region as $\|P\| = \max_{1\le k\le n}\{\Delta r_k, \Delta\theta_k\}$. In symbols,

$$\iint_R f(r,\theta)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k.$$

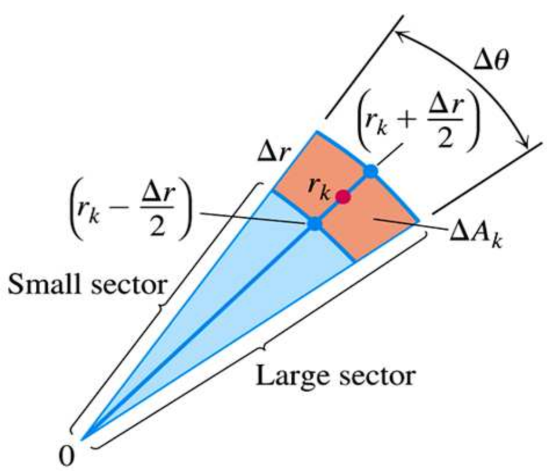

Figure 15.22, Page 872

Note. To evaluate the limit above, we need to evaluate ΔA_k in terms of Δr and $\Delta\theta$. We choose r_k to be the average of the radii of the inner and outer arcs bounding the kth polar rectangle ΔA_k. The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$. The radius of the outer arc is $r_k + (\Delta r/2)$. The area of a wedge-shaped sector of a circle having radius r and angle θ is $A = \dfrac{1}{2}\theta r^2$, as can be seen by multiplying πr^2, the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius: } \frac{1}{2}\left(r_k - \frac{\Delta r_k}{2}\right)^2 \Delta\theta$$

$$\text{Outer radius: } \frac{1}{2}\left(r_k + \frac{\Delta r_k}{2}\right)^2 \Delta\theta.$$

Therefore,

$$\begin{aligned}\Delta A_k &= \text{ area of large sector } - \text{ area of small sector}\\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right]\\ &= \frac{\Delta\theta}{2}(2r_k\Delta r) = r_k\,\Delta r\,\Delta\theta.\end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta.$$

As $\|P\| \to 0$, these sums converge to the double integral

$$\iint_R f(r,\theta)\, r\, dr\, d\theta.$$

Note. The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r,\theta)\, dA$ over a region R in polar coordinates, integrating first to r and then with respect to θ take the following steps.

1. *Sketch.* Sketch the region and label the bounding curves.

2. *Find the r-limits of integration.* Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.

3. *Find the θ-limits of integration.* Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.

Example. Page 876, number 6.

Definition. The *area* of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

Example. Page 876, number 28.

Note. The procedure for changing a Cartesian integral $\iint_R f(x,y)\,dx\,dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral. Then supply the polar limits of integration for the boundary R. The Cartesian integral then becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example. Page 876, number 10.

Examples. Page 876, numbers 38, 41.

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