Calculus 3

Chapter 15. Multiple Integrals

15.2. Double Integrals over General Regions—Examples and Proofs of Theorems

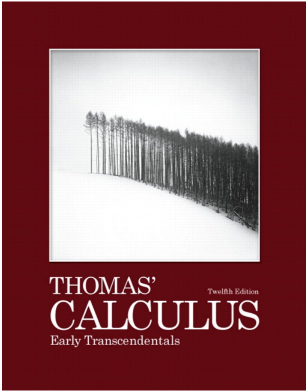

Table of contents

Exercise 15.2.20

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Solution. The region is:

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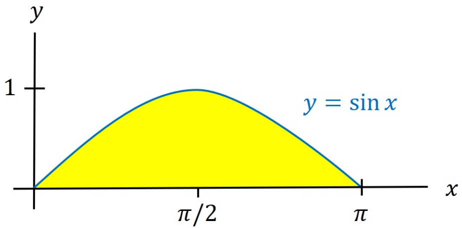

We evaluate the iterated integral as:

$$\int_0^{\pi} \int_0^{\sin x} y \, dy \, dx = \int_0^{\pi} \left. \frac{y^2}{2} \right|_{y=0}^{y=\sin x} dx = \int_0^{\pi} \frac{\sin^2 x}{2} - 0 \, dx$$

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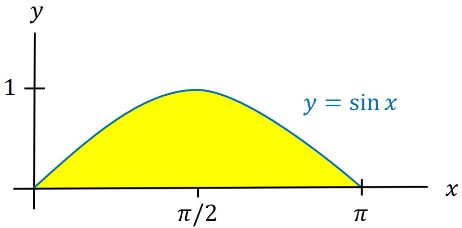

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Exercise 15.2.20 (continued)

Exercise 15.2.20. Sketch the region of integration and evaluate the double integral $\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$.

Solution (continued).

$$= \int_0^{\pi} \frac{1}{2} \frac{1 - \cos 2x}{2}\, dx \text{ since } \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{x}{4} - \frac{\sin 2x}{8} \Big|_{x=0}^{x=\pi} = \left(\frac{\pi}{4} - \frac{\sin 2\pi}{8} \right) - (0) = \frac{\pi}{4}.$$

□

Exercise 15.2.40

Exercise 15.2.40. Sketch the region of integration and write the equivalent double integral with the order of integration reverse:

$$\int_0^2 \int_0^{4-y^2} y \, dx \, dy.$$

Solution. We first have x ranging from 0 to $4 - y^2$, and second y ranges from 0 to 2. So the region is:

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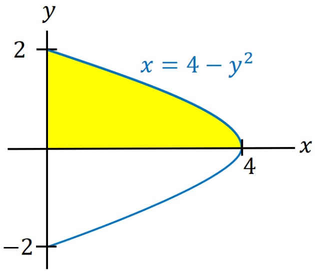

Now we can interpret that first y ranges from 0 to the curve $x = 4 - y^2$ (or $y = \sqrt{4-x}$, since $y \geq 0$ on the region) and second x ranges from 0 to 4. So the integral becomes

$$\int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx.$$

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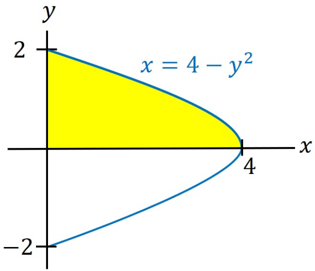

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Exercise 15.2.50

Exercise 15.2.50. Sketch the region of integration, reverse the order of integration, and evaluate the integral:

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx.$$

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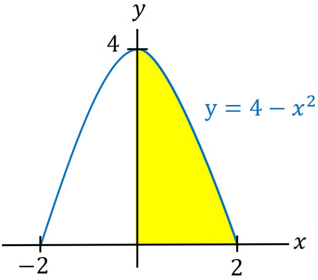

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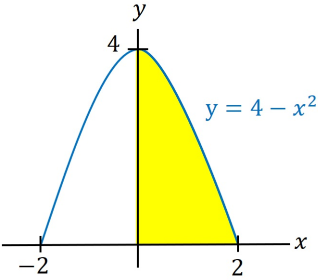

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Exercise 15.2.50 (continued)

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$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx.$$

Solution (continued). We now evaluate the new iterated integral:

$$\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy = \int_0^4 \left. \frac{x^2 e^{2y}}{2(4-y)} \right|_{x=0}^{x=\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{(\sqrt{4-y})^2 e^{2y}}{2(4-y)} - 0\, dy = \int_0^4 \frac{(4-y)e^{2y}}{2(4-y)}\, dy = \int_0^4 \frac{e^{2y}}{2}\, dy$$

$$= \left. \frac{e^{2y}}{4} \right|_{y=0}^{y=4} = \frac{e^{2(4)}}{4} - \frac{e^{2(0)}}{4} = \frac{e^8 - 1}{4}.$$

$\square$

Exercise 15.2.58

Exercise 15.2.58. Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

Solution. The region R is:

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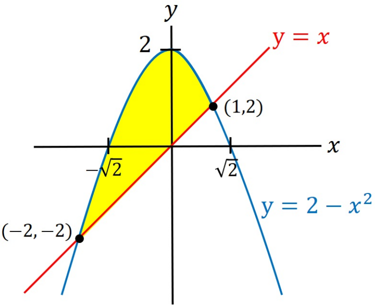

First y ranges from x to $2 - x^2$, and second x ranges from -2 to 1. Since $z = f(x, y) = x^2$ is nonnegative over R then the desired volume is

$$V = \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy \, dx.$$

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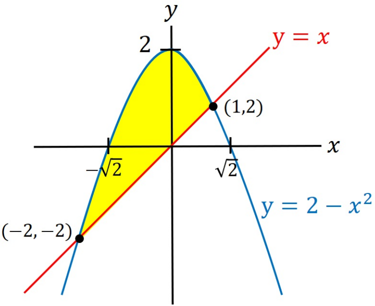

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Exercise 15.2.58 (continued)

Solution (continued). So the volume is

$$V = \int_{-2}^{1} \int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} x^2 y\Big|_{y=x}^{y=2-x^2}\,dx$$

$$= \int_{-2}^{1} x^2(2-x^2) - x^2(x)\,dx = \int_{-2}^{1} 2x^2 - x^4 - x^3\,dx = \frac{2x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4}\Big|_{x=-2}^{x=1}$$

$$= \left(\frac{2(1)^3}{3} - \frac{(1)^5}{5} - \frac{(1)^4}{4}\right) - \left(\frac{2(-2)^3}{3} - \frac{(-2)^5}{5} - \frac{(-2)^4}{4}\right)$$

$$= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} + \frac{16}{3} - \frac{32}{5} + 4 = \frac{40}{60} - \frac{12}{60} - \frac{15}{60} + \frac{320}{60} - \frac{384}{60} + \frac{240}{60} = \frac{189}{60} = \frac{63}{20}.$$

□

Exercise 15.2.76

Exercise 15.2.76. (Unbounded Region) Integrate $f(x,y) = \dfrac{1}{(x^2-x)(y-1)^{2/3}}$ over the infinite rectangle $2 \le x < \infty$, $0 \le y \le 2$.

Solution. We want to find $\displaystyle\int_2^\infty \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx$. This is an improper integral and so we write it as a limit:

$$\int_2^\infty \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx = \lim_{b\to\infty} \int_2^b \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x^2-x}\,\frac{(y-1)^{1/3}}{1/3}\Bigg|_{y=0}^{y=2} dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x^2-x}3((2)-1)^{1/3} - \frac{1}{x^2-x}3((0)-1)1/3\,dx$$

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Solution (continued).

$$= \lim_{b\to\infty} \int_2^b \frac{6}{x^2 - x}\, dx = 6 \lim_{b\to\infty} \int_2^b \frac{1}{x-1} - \frac{1}{x}\, dx \text{ by partial fractions}$$

$$= 6 \lim_{b\to\infty} (\ln(x-1) - \ln x)\Big|_{x=2}^{x=b} = 6 \lim_{b\to\infty} \ln\left(\frac{x-1}{x}\right)\Bigg|_{x=2}^{x=b}$$

$$= 6 \lim_{b\to\infty} \ln\left(\frac{b-1}{b}\right) - 6\ln\left(\frac{(2)-1}{2}\right) = -6\ln(1/2) = 6\ln 2.$$

□