Chapter 11. Parametric Equations and Polar Coordinates

11.2. Calculus with Parametric Curves

Definition. A parametrized curve $x = f(t)$ and $y = g(t)$ is *differentiable* at t if f and g are differentiable at t.

Note. By the Chain Rule, $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$, or $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ (assuming the three derivatives exist and $dx/dt \neq 0$). If $x = f(t)$ and $y = g(t)$ are twice-differentiable, then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}[y'] = \dfrac{dy'/dt}{dx/dt}$ where $y' = dy/dx$ (and, again, $dx/dt \neq 0$).

Example. Page 643, number 20.

Example. Page 643, number 22. HINT: In terms of dy-slices, the area is $\int_a^b x\,dy$.

Definition. Let C be a curve given parametrically by the equations $x = f(t)$ and $y = g(t)$ where $t \in [a, b]$. If f and g are continuously differentiable (that is, their derivatives are continuous $[a, b]$), then curve C is *smooth*.

Note. In Calculus 2 you saw that the length of a continuously differentiable function $y = f(x)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Informally, we can think of this as:

$$\begin{aligned}
L &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_{x=a}^{x=b} \sqrt{1 + (dy/dx)^2}\, dx \\
&= \int_{x=a}^{x=b} \sqrt{(1 + (dy/dx)^2)dx^2} = \int_{x=a}^{x=b} \sqrt{dx^2 + dy^2} \\
&= \int_{x=a}^{x=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_{x=a}^{x=b} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_{t_a}^{t_b} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt
\end{aligned}$$

where $f(t_a) = a$ and $f(t_b) = b$.

Definition. If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $t \in [a, b]$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the *length of* C is

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example. Page 643, number 26.

Note. In Calculus 2 you saw that the area of a surface of revolution which results from revolving $y = f(x)$ for $x \in [a, b]$ about the x-axis is $S = \int_a^b 2\pi y\, ds$ where ds is a differential of arclength. This inspires the following.

Definition. If a smooth curve $x = f(t)$, $y = g(t)$, for $t \in [a, b]$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x-axis $(y \geq 0)$**:**

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

2. Revolution about the y-axis $(x \geq 0)$**:**

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example. Page 644, number 47b.