Chapter 11. Parametric Equations and Polar Coordinates

11.6. Conic Sections

NOTE. Conic sections were first studied by the Greeks about 2300 years ago in connection with the three classic compass and straight-edge constructions: (1) trisection of an angle, (2) doubling the cube, and (3) squaring the circle. Conic sections result from intersecting a plane with two double cones. The three conic sections are the *ellipse* (a circle is a special case of an ellipse), the *parabola*, and the *hyperbola*.

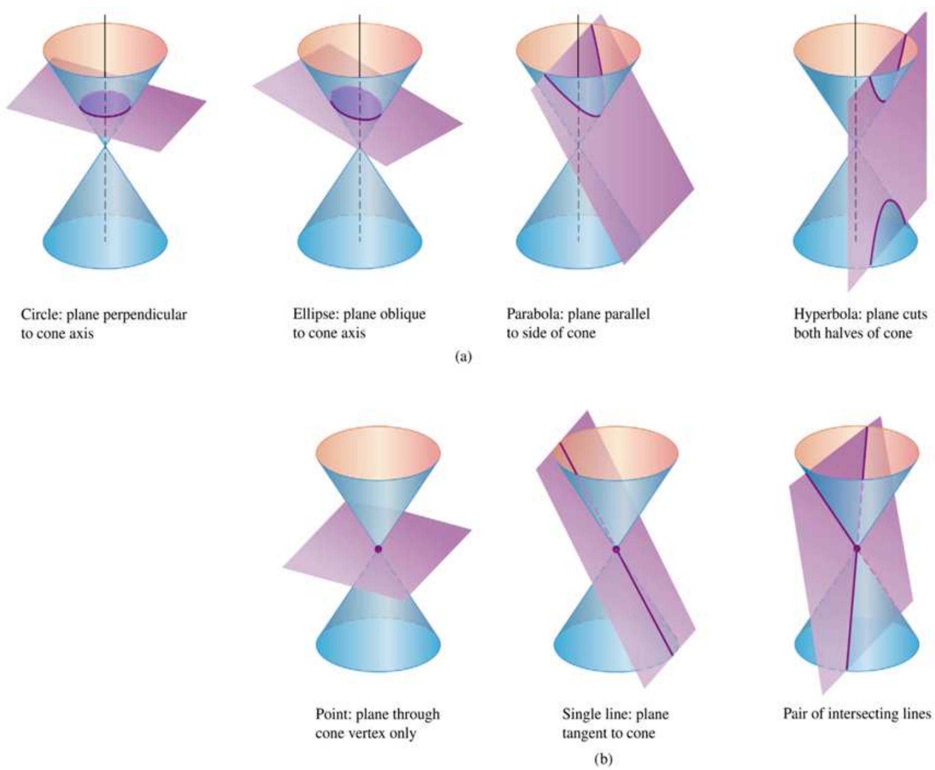

Figure 11.36, page 658

Definition. A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a *parabola*. The fixed point is the *focus* of the parabola. The fixed line is the *directrix*. The point on the parabola closest to the focus (and the directrix) is the *vertex*. The line through the vertex and focus is the *axis* and the distance from the vertex to the focus is the *focal length*.

Note. Suppose the focus of a parabola lies along the y-axis at the point $(F(0, p)$ and that the directrix is the line $y = -p$ (so that the vertex is the origin of the coordinate system). Let $P(x, y)$ be an arbitrary point on the parabola and let Q be the point on the directrix closest to P (so $Q = Q(x, -p))$:

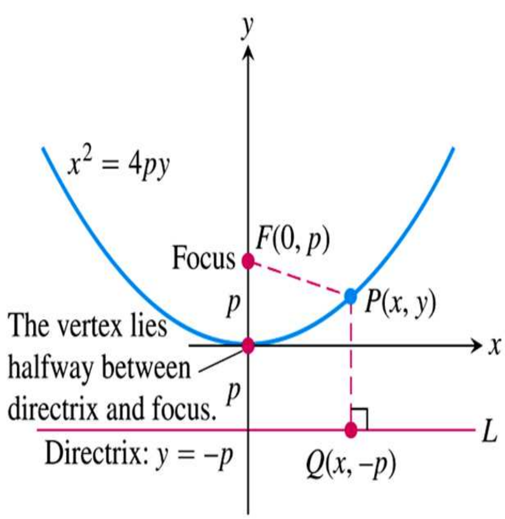

Figure 11.37, page 658

From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2} \text{ and}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

Since $PF = PQ$ we have

$$\sqrt{x^2 + (y-p)^2} = \sqrt{(y+p)^2} \text{ or}$$

$$x^2 + (y-p)^2 = (y+p)^2 \text{ or}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \text{ or}$$

$$x^2 = 4py \text{ or } y = \frac{x^2}{4p}.$$

If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then we get

$$x^2 = -4py \text{ or } y = -\frac{x^2}{4p}.$$

We can interchange x and y to get that an opening rightward parabola with vertex at the origin is of the form

$$y^2 = 4px \text{ or } x = \frac{y^2}{4p}$$

and an opening leftward parabola with vertex at the origin is of the form

$$y^2 = -4px \text{ or } x = -\frac{y^2}{4p}.$$

Example. Page 664, number 14.

Definition. An *ellipse* is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the *foci* of the ellipse. The line through the foci of an ellipse is the ellipse's *focal axis.* The point on the axis halfway between the foci is the *center.* The points where the focal axis and ellipse cross are the ellipse's *vertices.*

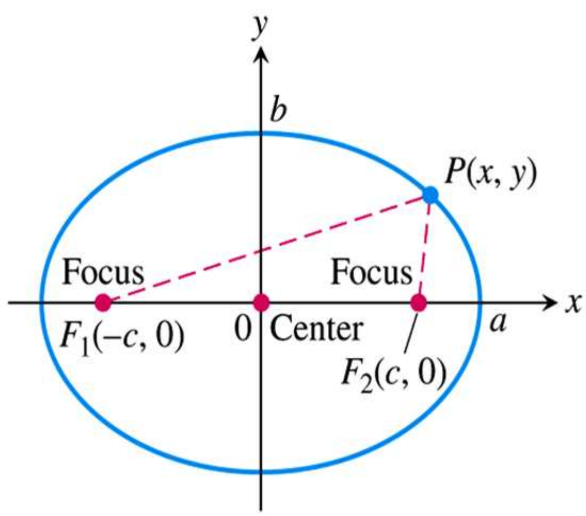

Figure 11.40, page 659

Note. Let the foci of an ellipse lie on the x-axis at $F_1(-c,0)$ and $F_2(c,0)$. Let $P(x,y)$ be an arbitrary point on the ellipse. Then we have that the distances PF_1 and PF_2 sum to a constant, say $PF_1 + PF2 = 2a$. Therefore we have

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a.$$

Manipulations give:

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2} \text{ or (squaring both sides)}$$

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \text{ or}$$

$$(x+c)^2 - 4a^2 - (x-c)^2 = -4a\sqrt{(x-c)^2+y^2} \text{ or}$$

$$x^2+2xc+c^2-4a^2-x^2+2xc-c^2 = -4a\sqrt{(x-c)^2+y^2} \text{ or}$$

$$4xc-4a^2 = -4a\sqrt{(x-c)^2+y^2} \text{ or (squaring both sides)}$$

$$16x^2c^2 - 32a^2xc + 16a^4 = 16a^2((x-c)^2+y^2) \text{ or}$$

$$\frac{x^2c^2}{a^2} - 2xc + a^2 = x^2 - 2xc + c^2 + y^2 \text{ or}$$

$$\frac{x^2c^2}{a^2} + a^2 = x^2 + c^2 + y^2 \text{ or}$$

$$x^2 - \frac{x^2c^2}{a^2} + y^2 = a^2 - c^2 \text{ or}$$

$$\frac{x^2a^2}{a^2} - \frac{x^2c^2}{a^2} + y^2 = a^2 - c^2 \text{ or}$$

$$\frac{x^2(a^2-c^2)}{a^2} + y^2 = a^2 - c^2 \text{ or}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1 \text{ or}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where we set $b^2 = a^2 - c^2$ in the last equation (notice from figure 11.40 that $a > c$ and so $a^2 - c^2$ is positive). The line segment in figure 11.40 joining the points $(\pm a, 0)$ is the *major axis*. The line segment joining the points

$(0, \pm b)$ is the *minor axis*. The distance a is the length of the *semimajor axis* and the distance $b = \sqrt{a^2 - c^2}$ is the *semiminor axis*. Notice that $c = \sqrt{a^2 - b^2}$ is the *center-to-focus distance*. We can similarly place the foci on the y-axis and get similar results.

Note. The *standard form equations for ellipses centered at the origin* are:

Foci on the x-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \ (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci $(\pm c, 0)$

Vertices $(\pm a, 0)$

Foci on the y-axis: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \ (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci $(0, \pm c)$

Vertices $(0, \pm a)$

In each case, a is the length of the semimajor axis and b is the length of the semiminor axis.

Example. Page 664, number 22.

Definition. A *hyperbola* is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the *foci* of the hyperbola. The line through the foci of a hyperbola is the *focal axis*. The point on the axis halfway between the foci is the hyperbola's *center*. The points where the focal axis and hyperbola cross are the *vertices* of the hyperbola.

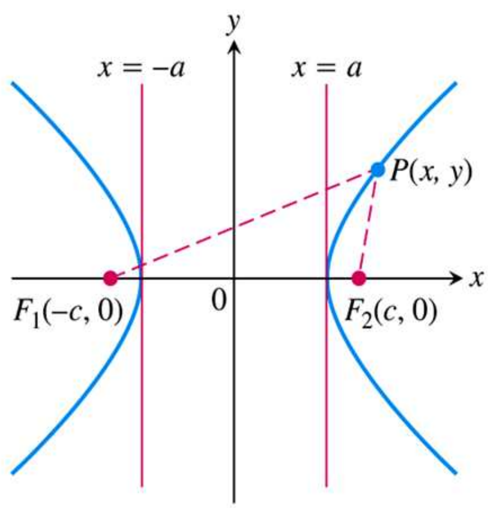

Figure 11.43, page 661

Note. Let the foci of a hyperbola be $F_1(-c, 0)$ and $F_2(c, 0)$. Let $P(x, y)$ be an arbitrary point on the hyperbola. Then we have that the distances PF_1 and PF_2 differ by a constant, say $PF_1 - PF2 = 2a$. Therefore we have

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Manipulations give:

$$\sqrt{(x+c)^2+y^2} = \pm 2a + \sqrt{(x-c)^2+y^2} \text{ or (squaring both sides)}$$

$$(x+c)^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \text{ or}$$

$$(x+c)^2 - 4a^2 - (x-c)^2 = \pm 4a\sqrt{(x-c)^2+y^2} \text{ or}$$

$$x^2+2xc+c^2-4a^2-x^2+2xc-c^2 = \pm 4a\sqrt{(x-c)^2+y^2} \text{ or}$$

$$4xc - 4a^2 = \pm 4a\sqrt{(x-c)^2+y^2} \text{ or (squaring both sides}$$

—the rest is like the computations for the ellipse)

$$\begin{aligned}
16x^2c^2 - 32a^2xc + 16a^4 &= 16a^2((x-c)^2+y^2) \text{ or}\\
\frac{x^2c^2}{a^2} - 2xc + a^2 &= x^2 - 2xc + c^2 + y^2 \text{ or}\\
\frac{x^2c^2}{a^2} + a^2 &= x^2 + c^2 + y^2 \text{ or}\\
x^2 - \frac{x^2c^2}{a^2} + y^2 &= a^2 - c^2 \text{ or}\\
\frac{x^2a^2}{a^2} - \frac{x^2c^2}{a^2} + y^2 &= a^2 - c^2 \text{ or}\\
\frac{x^2(a^2-c^2)}{a^2} + y^2 &= a^2 - c^2 \text{ or}\\
\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} &= 1 \text{ or}\\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}$$

where we set $-b^2 = a^2 - c^2$ in the last equation (notice from figure 11.44 that $a < c$ and so $a^2 - c^2$ is negative). This hyperbola has slant asymptotes of $y = \pm\frac{b}{a}x$. We can similarly place the foci on the y-axis and get similar results.

Note. The *standard form equations for hyperbolas centered at the origin* are:

Foci on the x-axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci $(\pm c, 0)$

Vertices $(\pm a, 0)$

Asymptotes $y = \pm\frac{b}{a}x$

Foci on the y-axis: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci $(0, \pm c)$

Vertices $(0, \pm a)$

Asymptotes $y = \pm\frac{a}{b}x$

Example. Page 664, number 28.

Note. Mirrors in flashlights, the headlights of your car, and the Hubble Space Telescope are in the shape of parabaloids (parabolas which are revolved about their axes). This is because a beam of light traveling parallel to the axis of a paraboloid-shaped mirror will bounce off the mirror and pass through the focus (and by placing a light bulb at the focus will result in the light rays traveling parallel to the axis). If a mirror is in the shape of an ellipse, then a ray of light emitted at one focus will travel to the other focus (echo rooms use this design). If a mirror is in the shape of a hyperbola, then a ray of light headed to the focus of one piece of the hyperbola will bounce off the mirror and go to the other focus (the secondary mirror of the Hubble Space Telescope is in the shape of a hyperboloid). These "optical properties" can be verified using calculus.

Example. Page 666, number 81 (the reflective property of parabolas).