Chapter 11. Parametric Equations and Polar Coordinates

11.7. Conic Sections in Polar Coordinates

Note. We start by introducing the *eccentricity* of a conic section. Quoting from the text (page 666): "The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and the degree to which it is 'squashed' or flattened."

Definition. The *eccentricity* of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a \geq b) \text{ is } e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The *eccentricity* of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The *eccentricity* of the parabola is 1.

Note. Notice that for an ellipse, $e \in [0, 1)$. The smaller the eccentricity of an ellipse the rounder it is (the eccentricity of a circle is 0) and larger the eccentricity of an ellipse, the more 'squashed' it is. The eccentricity of a hyperbola is greater than 1 and the larger the eccentricity of a hyperbola, the more "squashed" it is as well. For ellipses and hyperbolas, the eccentricity is the ratio of the distance between the foci to the distance between the vertices. All parabolas have eccentricity 1 and all are the same shape (though maybe different sizes).

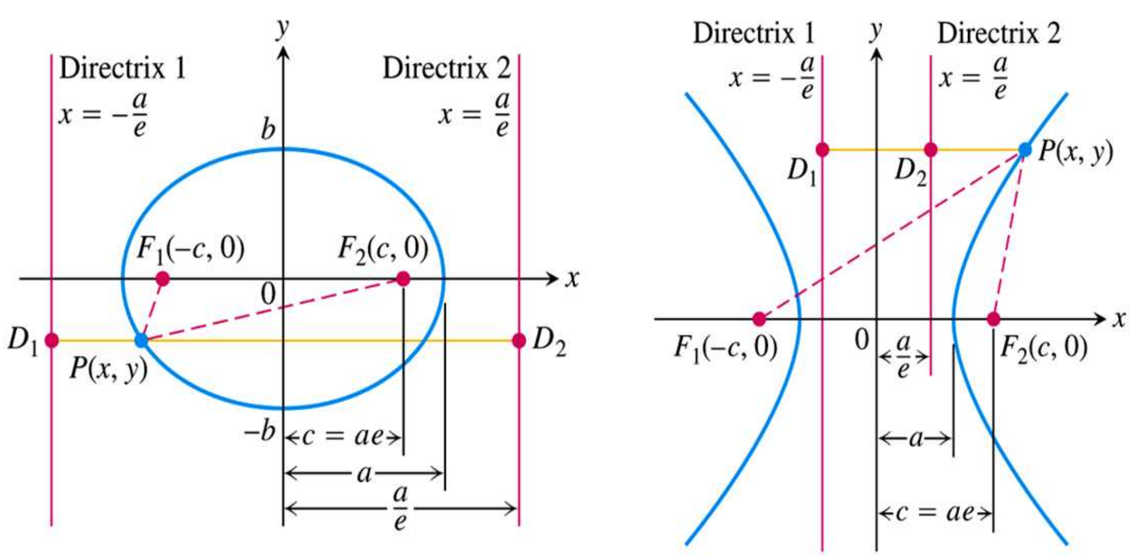
Examples. Page 671, numbers 8 and 26.

Note. We now seek a single formula which unifies all of the conic sections. (1) For a parabola, let P be an arbitrary point on the parabola, left F be the focus, and D be the point on the directrix closest to P. Then by the definition of *parabola*, we have the distance relationship $PF = 1 \cdot PD$. (2) For an ellipse, we define two directrices, the lines $x = \pm a/e$. Let P be an arbitrary point on the ellipse and let D_1 be the point on the directrix $x = -a/e$ closest to P and let D_2 be the point on the directrix $x = a/e$ closest to P. Then it can be shown that we have the distance relationships

$$PF_1 = e \cdot PD_1 \text{ and } PF_2 = e \cdot PD_2.$$

(3) For a hyperbola, we define two directrices, the lines $x = \pm a/e$. Let P be an arbitrary point on the hyperbola and let D_1 and D_2 be as they were for an ellipse. Then it can be shown that we have the distance relationships

$$PF_1 = e \cdot PD_1 \text{ and } PF_2 = e \cdot PD_2.$$



Figures 11.45 and 11.46, page 667

Definition. Let P be an arbitrary point on a conic section with eccentricity e. Then the *focus-directrix equation* for the conic is $PF = e \cdot PD$ where F is a focus of the conic and D is a point on a directrix closest to P.

Note. The focus-directrix equation will translate into Cartesian coordinates (x, y) in a way which depends on the value of e and will yield the three forms of equations given Section 11.6. However, in polar coordinates, the focus-directrix equation translates into a single form. Suppose a conic section has a focus at the origin O and directrix a vertical line $x = k$. Let P be an arbitrary point on the conic:

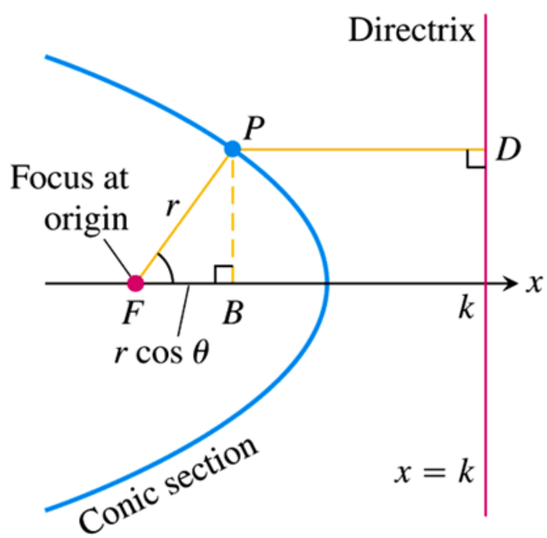


Figure 11.48, page 668

The polar coordinates of P satisfy $r = PF$ and $PD = k - FB = k - r\cos\theta$, where B is as given in the figure. The focus-directrix equation then implies that $PF = e \cdot PD$, or that $r = e(k - r\cos\theta)$.

Note. A conic section with one focus at the origin O, eccentricity e, and $x = k$ (where $k > 0$) as a vertical directrix has equation

$$r = \frac{ke}{1 + e\cos\theta}.$$

Another possible form for a conic with a focus at the origin is $r = \dfrac{ke}{1 - e\cos\theta}$. It is also possible for k to be negative. See page 669 for illustrations of different cases. For an ellipse where $k > 0$, we have $k = a/e - ea$ (see Figure 11.50 on page 669) and so we get the special equation for such an ellipse of

$$r = \frac{ke}{1 + e\cos\theta} = \frac{e(a/e - ea)}{1 + e\cos\theta} = \frac{a - e^2 a}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}.$$

Note. We now consider a line in polar coordinates. Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$. Let $P(r, \theta)$ be an arbitrary point on the line:

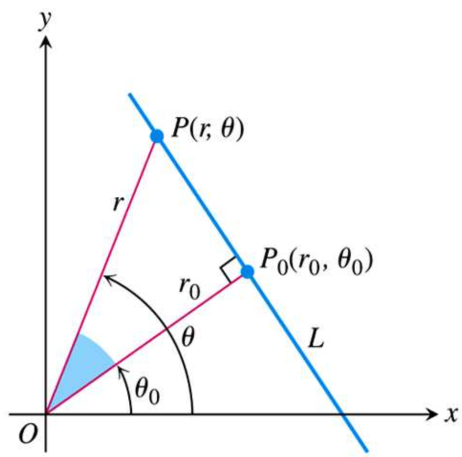


Figure 11.51, page 670

Since P, P_0, and O determine a right triangle, we have $r_0 = r\cos(\theta - \theta_0)$. We therefore have the formula of such a line in polar coordinates as:

$$r\cos(\theta - \theta_0) = r_0.$$

Example. Page 672, number 52.

Note. We now consider a circle in polar coordinates. Suppose the center of the circle is $P_0(r_0, \theta_0)$ and the radius of the circle is a:

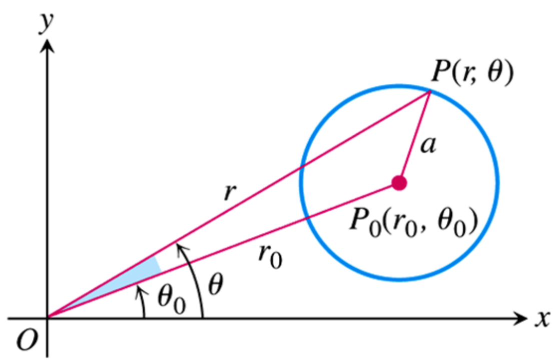


Figure 11.52, page 670

By the Law of Cosines for the triangle with vertices O, P, and P_0 we have

$$a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta - \theta_0).$$

Example. Page 672, number 62.

Examples. Page 672, numbers 75 and 76 (Earth).