Chapter 12. Vectors and the Geometry of Space

12.4. The Cross Product

Note. In this section we introduce an operation which can be performed on two vectors, each with three components. The operation is called *cross product*, or sometimes *vector product*. We use this product to find volumes and to construct planes in three dimensions.

Notes. Two non-parallel vectors **u** and **v** (i.e., **u** is not a scalar multiple of **v**) determine (in three-dimensions) a plane. If we make the tails of the vectors coincide, then we have three points: (1) the point at the tails of the vectors, (2) the head of vector **u**, and (3) the head of vector **v**. Three points determine a plane (just as two points determine a line). We now choose unit vector **n** to be perpendicular to both vectors **u** and **v** and so

that vectors **u**, **v**, and **n** form a right-hand coordinate system:
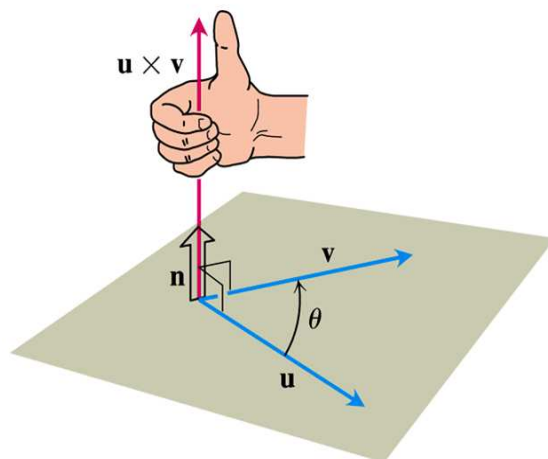

Figure 12.27, page 700

Definition. The *cross product* of two non-parallel vectors **u** and **v** is defined as

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}|\sin\theta)\mathbf{n},$$

where vector **n** is the unit vector mentioned above. If vectors **u** and **v** are parallel, then we define $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Theorem. Properties of the Cross Product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$ (Distribution of Scalar Multiplication over Cross Product).

2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (Distribution of Cross Product over Vector Addition).

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ (Anticommutivity of Cross Product).

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$ (Distribution of Cross Product over Vector Addition).

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$.

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Examples. Page 705, number 28c and 28g.

Note. This operation of taking a cross product of two vectors is likely the first operation you have encountered which is not commutative. Verification of the claims of this theorem are hard to establish, since we do not yet have a way to compute cross products (but we soon will). However, we can graphically convince ourselves of the validity of the following.

Theorem. The standard unit vectors satisfy the following cross product relationships:

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

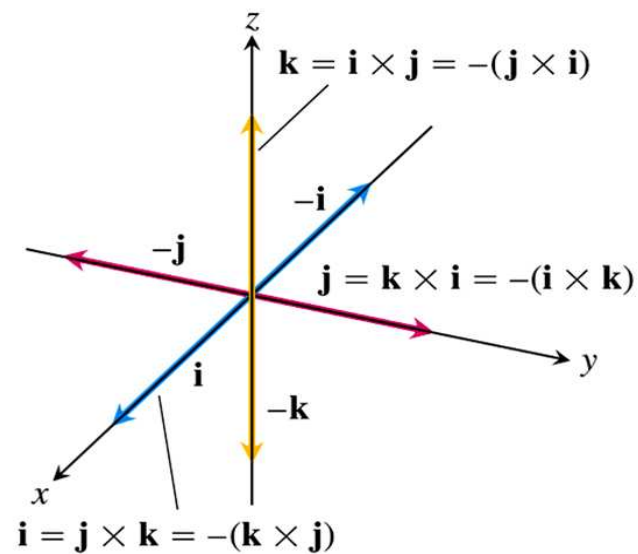$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

Figure 12.29, page 701

Note. We are now finally ready to compute $\mathbf{u} \times \mathbf{v}$ in terms of the components of $\mathbf{u}$ and $\mathbf{v}$. Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$. The computation is based on the cross products of the standard unit vectors and properties of cross product:

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\
&= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\
&= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.
\end{aligned}
$$

This computation can be remembered by computing what the book calls a *symbolic determinant* (determinants are explored in more detail in the Linear Algebra class):

$$
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}
$$

$$
= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.
$$

Example. Page 705, number 16b.

Note. We know that two non-parallel vectors, **u** and **v**, determine a plane. If we place these vectors with their tails at the same point, then we can use them to determine a parallelogram (see Figure 12.30 below). The area of this parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|.$$

We can use this fact to find the area of a triangle determined by three points.
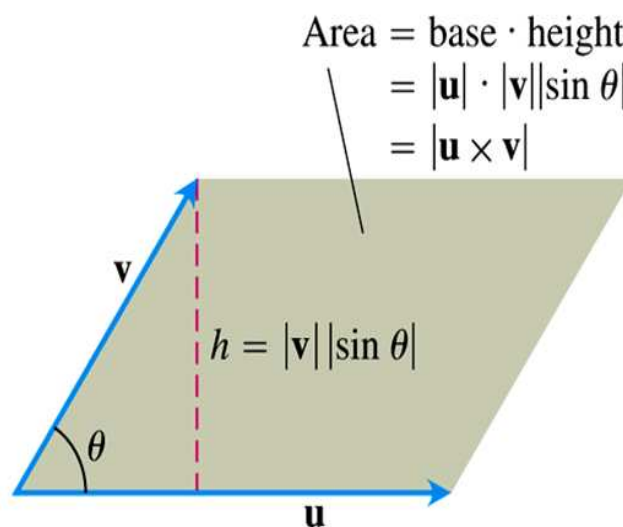

Figure 12.30, page 701

Example. Page 705, number 16a.

Note. Whenever you use a wrench, you exert a *torque* on the bolt being turned. Torque is a reflection of the force you exert and the length of the wrench (and so is measured, for example, in foot-pounds). If θ is the angle between the force and the wrench, then we find from Figure 12.32 that the torque is $|\mathbf{r}|\,|\mathbf{F}|\sin\theta$. Therefore we can compute the torque using cross product and we define it as a vector with direction $\mathbf{n}$ where the vectors $\mathbf{F}, \mathbf{r}, \mathbf{n}$ form a right hand coordinate system. We therefore have that:

$$\text{Magnitude of torque vector} = |\mathbf{r}|\,|\mathbf{F}|\sin\theta,$$

$$\text{Torque Vector} = (|\mathbf{r}|\,|\mathbf{F}|\sin\theta)\mathbf{n}.$$

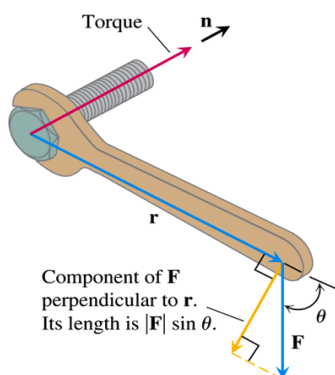Therefore the torque vector is $\mathbf{r} \times \mathbf{F}$.

Figure 12.32, page 703

Example. Page 705, number 26.

Definition. The *scalar triple product* of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in order) is

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| \, |\mathbf{w}| \, |\cos \theta|.$$

Note. We see from its definition that the scalar triple product represents the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
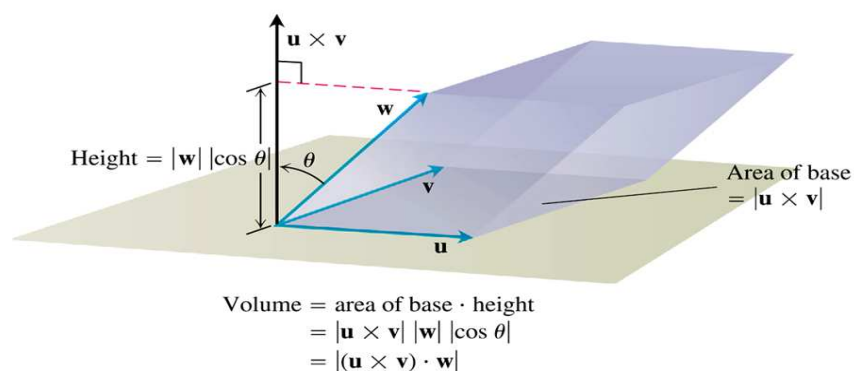

Figure 12.34, page 703

Note. We can verify computationally that:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \text{ and}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example. Page 705, number 22.