# Chapter 13. Vector-Valued Functions and Motion in Space

## 13.3. Arc Length in Space

**Definition.** The *length* of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, for $t \in [a, b]$, that is traced exactly once as $t$ increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\mathbf{v}|\, dt,$$

where $|\mathbf{v}| = |d\mathbf{r}/dt|$.

**Note.** If we choose a base point $P(t_0)$ on a smooth curve $C$ parametrized by $t$, each value of $t$ determines a point $P(t) = (x(t), y(t), z(t))$ on $C$ and a "directed distance

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)|\, d\tau = \int_{t_0}^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau,$$

measured along $C$ from the base point. We call $s$ an *arc length parameter* for the curve. In the next section, we will parametrize curves in terms of arclength in order to describe the shape of a curve in the sense of "curvature." The idea of curvature is to reflect how much a curve changes direction. This will be reflected in the acceleration vector, but only the component of the acceleration vector which reflects a change in direction.

This is why we will want to parametrize in terms of arc length, so as to traverse the curve at a uniform speed and hence the only acceleration will be in terms of change of direction.

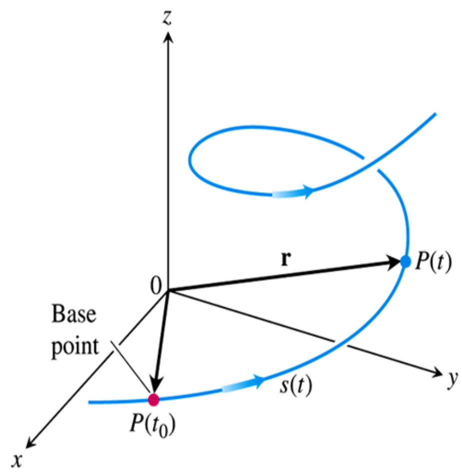


Figure 13.14, page 743

**Example.** Page 745, number 12.

**Definition.** The *unit tangent vector* to a curve $\mathbf{r}$ is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$.

**Example.** Page 745, number 6.