Chapter 13. Vector-Valued Functions and Motion in Space

13.5. Tangential and Normal Components of Acceleration

Note. If we let $\mathbf{r}(t)$ be a position function and interpret this as the movement of a particle as a function of time, then the unit tangent vector $\mathbf{T}$ represents the *direction* of travel of the particle and the principal unit vector $\mathbf{N}$ indicates the direction the path is turning into. Since both of these vectors are unit vectors, it is their *direction* that contains information. For any fixed time t, acceleration is a linear combination of $\mathbf{T}$ and $\mathbf{N}$: $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for some a_T and a_N.

Definition. Define the *unit binormal vector* as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Note. Notice that since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, then $\mathbf{B}$ is in fact a unit vector. Changes in vector $\mathbf{B}$ reflect the tendency of the motion of the particle with position function $\mathbf{r}(t)$ to 'twist' out of the plane created by vectors $\mathbf{T}$ and $\mathbf{N}$. Also notice that vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ define a moving right-hand vector "frame." This frame is called the

Frenet frame or the **TNB** *frame*.

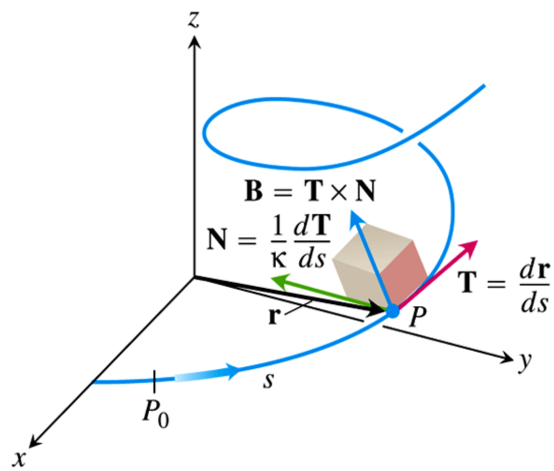

Figure 13.23, page 752

Note. As commented above, we can write $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for some a_T and a_N. We want to find formulae for a_T and a_N. By the Chain Rule,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

So acceleration is

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\mathbf{T}\frac{ds}{dt}\right] = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}. \end{aligned}$$

(Recall that $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.)

Definition. If the acceleration vector is written as $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, then

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}[|\mathbf{v}|] \text{ and } a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2$$

are the *tangential* and *normal* scalar components of acceleration. (Recall that s is arclength and so ds/dt is the rate at which arclength is traversed with respect to time. That is, ds/dt is speed: $ds/dt = |\mathbf{v}|$.)

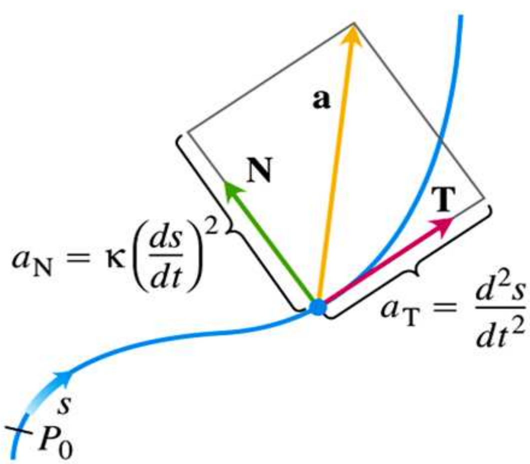

Figure 13.25, page 753

Note. If we are given the position function $\mathbf{r}(t)$, then a_T is easy to find (just calculate $\frac{d}{dt}\left[\left|\frac{d\mathbf{r}}{dt}\right|\right]$). But the computation of a_N seems to require us to find curvature κ. But there is a quicker way. Since $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and $\mathbf{T}$ and $\mathbf{N}$ are orthogonal, then $|\mathbf{a}|^2 = a_T^2 + a_N^2$. Therefore we can solve to a_N and find that: $a_N = \sqrt{|\mathbf{a}|^2 - a_T}$.

Example. Page 756, number 8.

Note. We have commented that changes in the binormal vector $\mathbf{B}$ reflect the tendency of the motion of the particle with position function $\mathbf{r}(t)$ to 'twist' out of the plane created by vectors $\mathbf{T}$ and $\mathbf{N}$. This twisting is called *torsion*. We are interested in how $\mathbf{B}$ changes with respect to arclength s:

$$\frac{d\mathbf{B}}{ds} = \frac{d[\mathbf{T}\times\mathbf{N}]}{ds} = \frac{\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{0} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

since $d\mathbf{T}/ds$ is parallel to $\mathbf{N}$.

We need a quick result concerning vector functions of constant magnitude (see page 731): **Lemma.** If $\mathbf{r}(t)$ is a vector function such that $|\mathbf{r}(t)| = c$ for some constant c, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. The proof is computational:

$$\begin{aligned}
\mathbf{r}(t)\cdot\mathbf{r}(t) &= |\mathbf{r}(t)|^2 = c^2 \\
\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] &= \frac{d}{dt}[c^2] \\
\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\
2\mathbf{r}'(t)\cdot\mathbf{r}(t) &= 0.
\end{aligned}$$

Since $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, the vectors are orthogonal.

Returning to $\mathbf{B}$, we know from above that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ since it is the cross product of vector $\mathbf{T}$ and another vector. Since $\mathbf{B}$

is always a unit vector, then by Lemma $d\mathbf{B}/ds$ is also orthogonal to $\mathbf{B}$. Therefore $d\mathbf{B}/ds$ must be a multiple of vector $\mathbf{N}$. We define the *torsion* τ with the formula $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. We can compute τ as follows:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau$$

and so $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N}$. As the book states, the curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. The torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve. "Torsion measures how the curve twists. ...In a more advanced course it can be shown that a space curve is a helix if and only if it has constant nonzero curvature and constant nonzero torsion." [Smiley Face!]

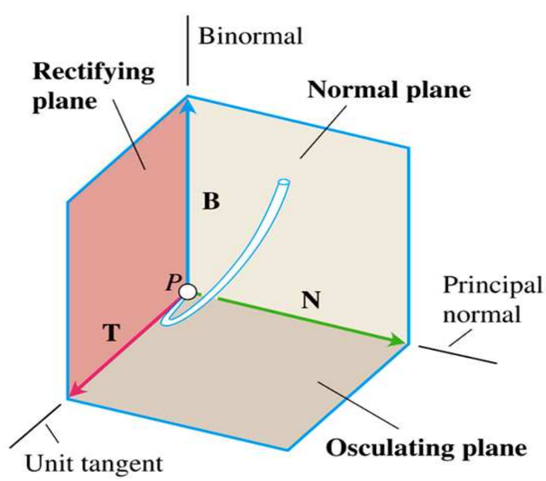

Figure 13.28, page 755

Note. Consider a position function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. It can be shown ("in more advanced texts") that torsion can be computed as

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

where $\mathbf{v} \times \mathbf{a} \neq \mathbf{0}$ and the dots indicate (as is tradition in physics) derivatives with respect to time t: $\dot{x} = dx/dt$. So the first row of the matrix consists of the components of velocity $\mathbf{r}'(t) = \mathbf{v}$, the second row consists of components of acceleration $\mathbf{r}''(t) = \mathbf{a}$ and the third row consists of components of jerk $\mathbf{r}'''(t)$.

Examples. Page 757, numbers 14 and 26.

Note. In summary, we have the following formulae:

Position: $\mathbf{r}(t) = \mathbf{r}$

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$ (see Page 756, number 21)

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = -\dfrac{1}{|\mathbf{v}|}\left(\dfrac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right)$ (see Page 757, number 28)

Tangential and normal scalar components of acceleration:

$$\mathbf{A} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = \frac{d}{dt}\left[|\mathbf{v}|\right]$ and $a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}^2 - a_T}$.

Note. For an alternate treatment of this same material, see Section 1-1 of my notes for *Differential Geometry* (MATH 5510) at:

`http://faculty.etsu.edu/gardnerr/5310/notes.htm`.

Section 1-2 of these notes deals with the curvature of surfaces.