Chapter 14. Partial Derivatives

14.2. Limits and Continuity in Higher Dimensions

Note. Analogous to the behavior of a function of a single variable, we wish to cleanly define the concept of *limit* for a function of "several" variables (in this section "several" means two, but the ideas are easily extended to more than two variables). If the values of $f(x, y)$ lie *arbitrarily close* to a fixed real number L for all points (x, y) *sufficiently close* to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). As in Calculus 1, we just need to clearly define the "arbitrarily/sufficiently" stuff. However, the textbook somewhat deviates from the definition of limit from Calculus 1 and this has some weird consequences!

Thomas' Definition. We say that a function $f(x, y)$ approaches the *limit* L as (x, y) approaches (x_0, y_0), denoted $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

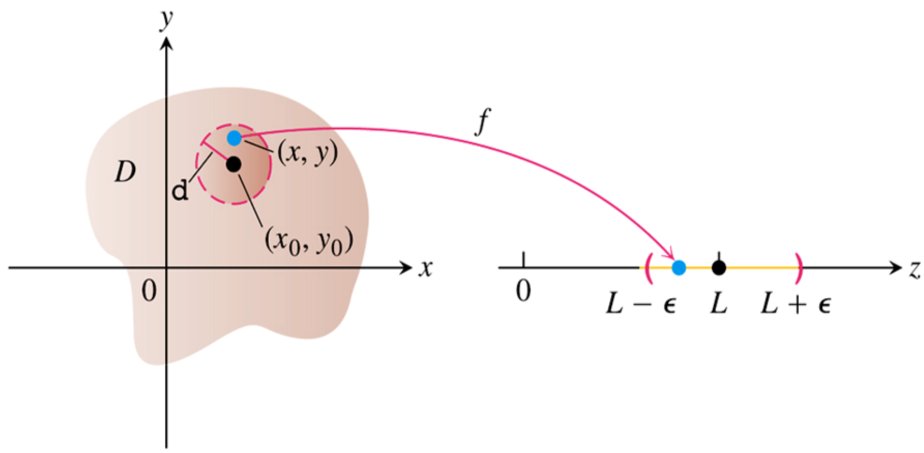

Figure 14.12, page 774 (the "d" here should be a δ—this is a typo in this figure, though your text has this correctly labeled).

Note. Notice the restriction of consideration to points (x, y) **in the domain of** f!!! This is different from the definition of $\lim_{x \to x_0} f(x) = L$ on page 77 where it is required that the function "$f(x)$ be defined on an open interval containing x_0 except possibly at x_0 itself." So in Calculus 1, you saw that $\lim_{x \to 0} \sqrt{x}$ does not exist (since the corresponding left-sided limit does not exist—it's a square-root-of-negatives problem). However, in the current setting of section 14.2, we would ignore any square roots of negatives since any points (x, y) which would generate this are not in the domain of the function. Therefore, we have the following result which we will prove from the definition of limit: $\lim_{(x,y)\to(0,0)} \sqrt{x} = 0$. This is in seeming contradiction to the fact that $\lim_{x\to 0} \sqrt{x}$ does not exist, but this strange situation arises from the fact that the textbook is treating limits in a rather fundamentally different way in this section (as it also did in section 13.1). More soon, but first an example.

Example. Use the definition of limit to prove that $\lim_{(x,y)\to(0,0)} \sqrt{x} = 0$.

Proof. Let $\epsilon > 0$ be an arbitrary number. Then we need to find a corresponding number $\delta > 0$ which will satisfy the definition of limit given above. We choose (omitting the details on *why* we make this choice) $\delta = \epsilon^2$. Consider (x, y) in the domain of $f(x) = \sqrt{x}$ (the domain of f is $\{(x, y) \mid x \geq 0\}$) such that $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} < \delta$. Notice that $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$ and so this implies that $|x| < \delta = \epsilon^2$. Since we only consider (x, y) in the domain of f, we have $0 \leq x < \epsilon^2$. Therefore $\sqrt{x} < \sqrt{\epsilon^2} = |\epsilon| = \epsilon$. Hence, we have $\sqrt{x} = |\sqrt{x}| = |\sqrt{x} - 0| = |f(x, y) - L| \leq \epsilon$. Therefore the definition of limit is satisfied and we conclude that $\lim_{(x,y)\to(0,0)} \sqrt{x} = 0$.

Q.E.D.

Note. Since there is no restriction in Thomas' Definition on the relationship between the domain of f and the point (x_0, y_0) (such as having f be defined "near" (x_0, y_0)), then we can get some totally bizarre results. Both of the following are true statements (here we are, again, dealing with **bonus education**):

$$\lim_{(x,y)\to(-1,-1)} \sqrt{x}\sqrt{y} = 5 \text{ and } \lim_{(x,y)\to(-1,-1)} \sqrt{x}\sqrt{y} = 7.$$

In fact, we can accurately say (given Thomas' Definition) that

$$\lim_{(x,y)\to(-1,-1)} \sqrt{x}\sqrt{y}$$

equals any value you like! This is a bit of a logical trick (something the text should be more careful of avoiding!) and works like this. Let $\epsilon > 0$. Choose $\delta = 1$. Then for any point (x, y) in the domain of f satisfying $0 < \sqrt{(x-(-1))^2 + (y-(-1))^2} = \sqrt{(x+1)^2 + (y+1)^2} < \delta = 1$ (of which there are **no such points**!), we have $|f(x, y) - L| < \epsilon$ (where we can take L to be 5, 7, or anything). The logical trick is that the book's definition is *vacuously* satisfied—it is true that all such points (x, y) satisfy this relationship since there are no such points! This may seem like mathematical sorcery, but **we can't let this stand!!!** One solution is to require that the function be defined "close to" point (x_0, y_0).

Alternate Definition 1. Let $f(x, y)$ be defined on a disk centered at (x_0, y_0), except possibly at (x_0, y_0) itself. We say that a function $f(x, y)$ approaches the *limit* L as (x, y) approaches (x_0, y_0), denoted $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y)

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Note. This definition eliminates the weird behavior described above where a limit can have more than one value. It also is consistent with the definition of limit of a function of a single variable given on page 77. However, this definition is somewhat restrictive, and would not allow us to say the limit in Example 2 on page 775 exists. A better way to deal with this is the following.

Alternate Definition 2. Let (x_0, y_0) be a limit point of the domain of f. We say that a function $f(x, y)$ approaches the *limit* L as (x, y) approaches (x_0, y_0), denoted $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Note. This definition eliminates the weird behavior described above where a limit can have more than one value. However, it still keeps $\lim_{(x,y)\to(0,0)} \sqrt{x} = 0$. The best way (i.e., the most practical way) for us to deal with this is to take Alternate Definition 2 as our definition of limit for a function of two variables and to view the facts that

$$\lim_{(x,y)\to(0,0)} \sqrt{x} = 0 \text{ and } \lim_{x\to 0} \sqrt{x} \text{ does not exist}$$

as the result of considering similar questions, but in *different settings* (namely, functions of a single variable versus functions of two variables). We could attain the highest level of consistency and diversity of application, by using Alternate Definition 1 and revising the definition of limit of a function of a single variable to require x_0 to be a limit point of the domain of the function and to only consider points in the domain of the function:

Proposed Alternate Definition to That Given on Page 77.

Let x_0 be a limit point of the domain of f. We say that a function $f(x)$ approaches the *limit* L as x approaches x_0, denoted $\lim_{x\to x_0} f(x) = L$, if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x in the domain of f,

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - x_0| < \delta.$$

Under this definition, $\lim_{x \to 0} \sqrt{x} = 0$ (a result you might find pleasing, since it can be evaluated with substitution). In fact, some texts (usually more advanced than a calculus text) take this as the definition of limit. So **enough to the bonus education**, and back to the task at hand.

Theorem 1. Properties of Limits of Functions of Two Variables.

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$

2. *Difference Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$

3. *Constant Multiple Rule*: $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)

4. *Product Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)g(x,y)) = LM$

5. *Quotient Rule*: $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$

6. *Power Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$, n a positive integer

7. *Root Rule*: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer and if n is even, we assume $L \geq 0$.

Note. The textbook makes a bit of an error here. In the Root Rule, the book state that it requires $L > 0$ when n is even. However, with the book's definition of limit (as well as with our Alternate Definition 2) we can also allow $L = 0$. Were we to take Alternate Definition 1, then we would need the strict inequality $L > 0$. Under Thomas' Definition of limit of a function of a single variable on page 77, the Root Rule only holds for $L > 0$ when n is even (see page 68). All of this is the result of whether or not we consider only values of the independent variable(s) which are in the domain or not and the issue of square roots of negatives (an issue which potentially arises when n is even and $L = 0$). A funny story is how the 9th and 10th editions of *Thomas' Calculus* mistakenly allowed $L = 0$ in the Root Rule when considering limits of functions of a single variable…ask me about it sometime…

Example. Page 775, Example 2. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$. Notice that any point (x, y) where $x = y$ is not in the domain of the function and $(x - y) \neq 0$.

Example. Page 780, number 20. Notice the textbook's restriction of $x \neq y + 1$. It is unnecessary to state this under the book's definition of limit (and under our Alternative Definition 2) since any point (x, y) where $x = y + 1$ is not in the domain of the function.

Definition. A function $f(x, y)$ is *continuous at the point* (x_0, y_0) if

1. f is defined at (x_0, y_0),

2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and

3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is *continuous* if it is continuous at every point of its domain.

Note. And again, since the textbook's definition of limit in this chapter is different from the definition of Chapter 2, then continuity is slightly different here than in Chapter 2. Compare the definition of continuity here to that on page 94 (the definition of continuity of a function of a single variable at an *interior point* of its domain).

Example. Page 780, number 32a.

Note. To actually evaluate limits, we can use Theorem 1, along with the standard "factor, cancel, substitute" ("FCS") method. However, it can be difficult to establish that a particular limit *does not* exist. In Calculus 1, you could test left-hand and right-hand limits to see if the "regular" two-sided limit exists. However, if a function consists of two (or more) variables, then there are an infinite number of directions from which we can approach a point (x_0, y_0). We probably cannot test *all* of these directions to see if they are the same, but we can cleverly check two of them to see if they are *different*. That's the idea behind the following.

Theorem. Two-Path Test for Nonexistence of a Limit.

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist. (NOTE: You'll be relieved to hear that this holds regardless of which of the many possible definitions we take of limit!)

Example. Page 780, number 46.

Theorem. Continuity of Composites.

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h(x, y) = g(f(x, y)) = g \circ f$ is continuous at (x_0, y_0).

Example. Page 780, number 40.