Chapter 14. Partial Derivatives

14.5. Directional Derivatives and Gradient Vectors

Note. The following figure gives level curves of land in New York near the Hudson River. Notice how the streams (in blue) always intersect the level curves at right angles. This is due to the fact that the streams are following paths of steepest descent are are going downhill as fast as possible.

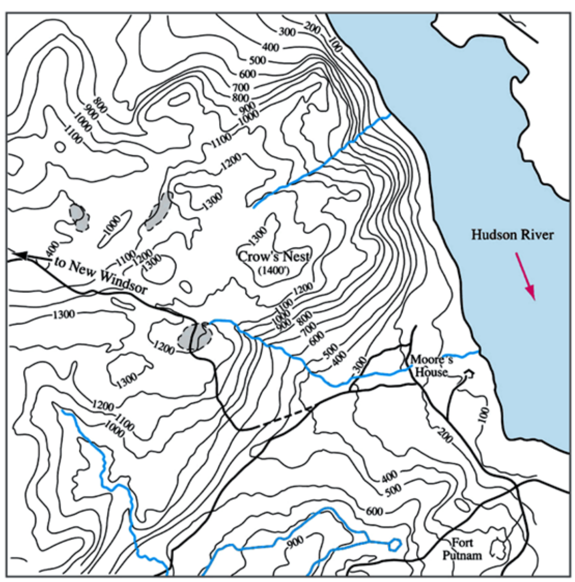

Figure 14.25, Page 802

Definition. The *derivative* of $f(x, y)$ at point $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} = (D_{\mathbf{u}}f)P_0,$$

provided the limit exists.

Note. df/ds as given above is the rate of change of f at P_0 in the direction $\mathbf{u}$.

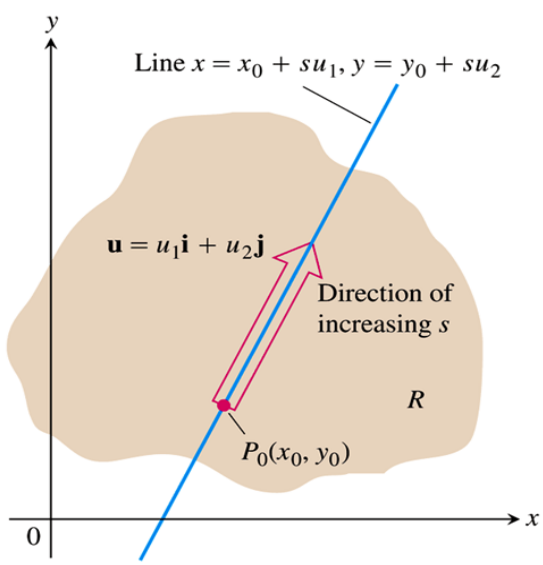

Figure 14.26, Page 802

Note. The equation $x = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0, 0)$ parallel to $\mathbf{u}$ intersects S in a curve C. The rate of change of f in the direction $\mathbf{u}$ is the slope of the tangent to C at P. When $\mathbf{u} = \mathbf{i}$, the directional derivative at P_0 is $\partial f/\partial x$ evaluated at (x_0, y_0). Then $\mathbf{u} = \mathbf{j}$, the directional derivative at P_0 is $\partial f/\partial y$ evaluated at (x_0, y_0).

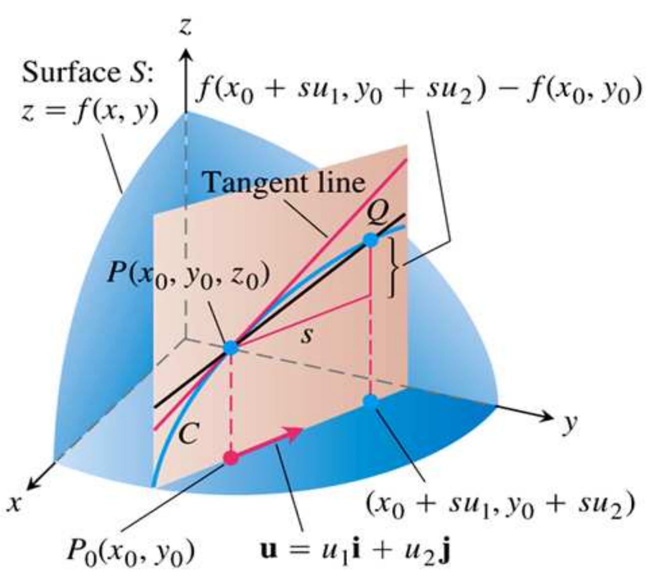

Figure 14.27, Page 803

Note. We now need an easy way to calculate directional derivatives without using the limit definition. Consider the parametric line $x = x_0 + su_1$, $y = y_0 + su_2$ through point $P_0(x_0, y_0)$, parametrized with respect to arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then by the Chain Rule we find:

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\left[\frac{dx}{ds}\right] + \left(\frac{\partial f}{\partial y}\right)_{P_0}\left[\frac{dy}{ds}\right] \\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0}[u_1] + \left(\frac{\partial f}{\partial y}\right)_{P_0}[u_2] \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]\cdot[u_1\mathbf{i} + u_2\mathbf{j}].
\end{aligned}$$

Definition. The *gradient vector* (*gradient*) of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained be evaluating the partial derivatives of f at P_0.

Example. Page 808, number 4.

Theorem 9. The Directional Derivative Is a Dot Product.

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u},$$

the dot product of the gradient ∇f at P_0 and unit vector $\mathbf{u}$.

Example. Page 808, number 18.

Theorem 14.5.A. Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P of its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|$.

2. Similarly, f decreases most rapidly in the direction $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.

3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and $D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| 0 = 0$.

Example. Page 809, number 22.

Theorem 14.5.B. At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Proof. If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$, then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\begin{aligned}
\frac{d}{dt}[f(g(t), h(t)) &= \frac{d}{dt}[c] \\
\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{d}{dt} &= 0 \\
\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right) &= 0 \\
\nabla f \cdot \frac{d\mathbf{r}}{dt} &= 0.
\end{aligned}$$

Since $d\mathbf{r}/dt$ is tangent to the level curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ and $\nabla f \cdot \dfrac{d\mathbf{r}}{dt} = 0$, then the gradient of f is normal to the level curve. *Q.E.D.*

Note. The previous theorem allows us to find equations for tangent lines to level curves. They are lines normal to the gradients. The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0.$$

If $\mathbf{N}$ is the gradient $(\nabla f)_{x_0,y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, the equation for the tangent line is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Example. Page 809, number 26.

Theorem 14.5.C. Algebraic Rules for Gradients

1. *Sum Rule*: $\nabla(f + g) = \nabla f + \nabla g$

2. *Difference Rule*: $\nabla(f - g) = \nabla f - \nabla g$

3. *Constant Multiple Rule*: $\nabla(kf) = k\nabla f$ (for any number k)

4. *Product Rule*: $\nabla(fg) = f\nabla g + g\nabla f$

5. *Quotient Rule*: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Example. Page 809, number 40(d).

Note. For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

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