Chapter 14. Partial Derivatives

14.6. Tangent Planes and Differentials

Note. If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t gives

$$\begin{aligned}
\frac{d}{dt}[f(g(t), h(t), k(t)) &= \frac{d}{dt}[c] \\
\frac{\partial f}{\partial x}\overset{\curvearrowright}{\left[\frac{dg}{dt}\right]} + \frac{\partial f}{\partial y}\overset{\curvearrowright}{\left[\frac{dh}{dt}\right]} + \frac{\partial f}{\partial z}\overset{\curvearrowright}{\left[\frac{dk}{dt}\right]} &= 0 \\
\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j} + \frac{dk}{dt}\mathbf{k}\right) &= 0 \\
(\nabla f) \cdot \left(\frac{d\mathbf{r}}{dt}\right) &= 0.
\end{aligned}$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector. In the figure below, we see that all the velocity vectors at point P_0 are orthogonal to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to ∇f. Therefore, the gradient of f at P_0 will act as a normal vector to the tangent plane to the surface at P_0.

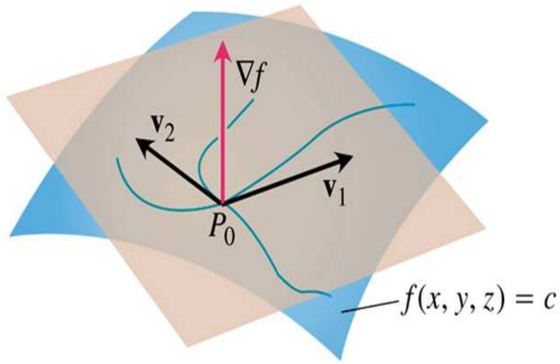

Figure 14.32, Page 810

Definition. The *tangent plane* at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. The *normal line* of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Note. The equation of the tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

The equation of the normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P_0)t, \;\; y = y_0 + f_y(P_0)t, \;\; z = z_0 + f_z(P_0)t.$$

Example. Page 817, number 4.

Note. If we consider the function $z = f(x, y)$, then the tangent plane to this surface at the point $(x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example. Page 817, number 10.

Note. We now use differentials to estimate changes in functions, similar to what was done for functions of a single variable in section 3.11. To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, we use the differential

$$df = \left(\nabla f|_{P_0} \cdot \mathbf{u} \right) ds.$$

Notice that df is the directional derivative of f times the distance increment ds.

Example. Page 817, number 20.

Definition. The *linearization* of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is the *standard linear approximation* of f at (x_0, y_0).

Note. In fact, the plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0) (just as the line $y = L(x)$ was the tangent line to $y = f(x)$ at the point of approximation in section 3.11). Thus, the linearization of a function of two variables is a tangent-plane approximation. As long as (x, y) is "close to" (x_0, y_0) (that is, if Δx and Δy are small), then $L(x, y)$ will take on approximately the same values as $f(x, y)$.

Example. Page 818, number 30.

Note. If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by it linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x,y)| \leq \frac{1}{2}M(|x-x_0|+|y-y_0|)^2.$$

Notice that the error is small when M, Δx, and/or Δy are small (especially Δx and Δy).

Example. Page 818, number 50.

Definition. The *differentials* dx and dy are independent variables (so they can take on any values). Often we take $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$. If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

in the linearization of f is called the *total differential* of f.

Example. Page 819, number 52.

Note. We can extend the ideas of this section to functions of more than two variables. For functions of three variables, we have the following.

1. The *linearization* of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the error $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by "small" amounts dx, dy, and dz, the total differential

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

gives a "good" approximation of the resulting change in f.

Example. Page 818, number 44.