Chapter 14. Partial Derivatives

14.8. Lagrange Multipliers

Note. In this section, we consider extrema of functions of several (well, two) variables where the is an added constraint (i.e., relationship) between the two variables. We start with an example.

Example. Example 2, page 830. Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

Solution. These are the points whose coordinates minimize the value of the functions $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint that $x^2 - z^2 - 1 = 0$. Notice that the function $f(x, y, z)$ has as its level curves spheres of various radii centered at the origin. So we geometrically consider a small sphere centered at the origin which expands. At the instant when the sphere contacts the hyperbolic cylinder, both surfaces will have the same tangent plane and normal line at the points of contact.

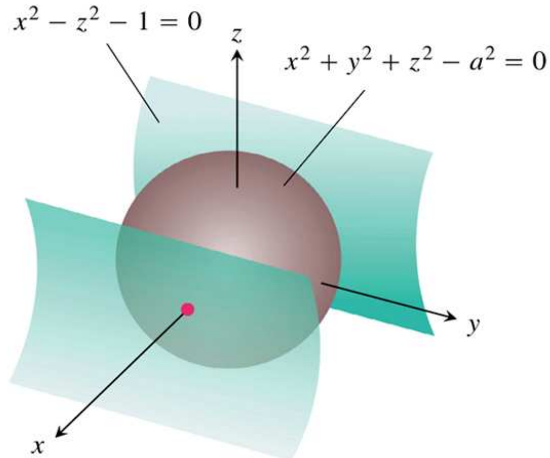

Figure 14.51, Page 831

Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \text{ and } g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact, we should be able to find a scalar λ such that $\nabla f = \lambda \nabla g$, or in this case, $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2z\mathbf{k})$. Solving, we find (see the text for details) that the desired points are $(-1, 0, 0)$ and $(1, 0, 0)$.

Note. The method used above is the method of *Lagrange multipliers*. It implies that the extreme values of function $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are to be found on the surface $g = 0$ among the points where $\nabla f = \lambda \nabla g$ for some scalar λ (called the *Lagrange multiplier*).

Theorem 12. The Orthogonal Gradient Theorem. Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C : \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof. The values of f on C are given by the composite $f(g(t), h(t), k(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = \nabla f \cdot \mathbf{v}.$$

At any point P_0 where f has a local extrema relative to its values on the curve, we have $df/dt = 0$ and so $\nabla f \cdot \mathbf{v} = 0$. *Q.E.D.*

Note. Theorem 12 is the key to the method of Lagrange multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that point P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface. We assume also that $\nabla g \neq \mathbf{0}$ at the points on the surface $g(x, y, z) = 0$. Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$. Therefore, ∇f is orthogonal to the velocity vector of every such differentiable curve through P_0. So is ∇g, since ∇g is orthogonal to the level surface $g = 0$. Therefore, at P_0, ∇f is some scalar multiple λ of ∇g. In summary:

The Method of Lagrange Multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. to find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$. For functions of two independent variables, the condition is similar, without the variable z.

Example. Page 837, number 30.

Note. Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \text{ and } g_2(x, y, z) = 0$$

and g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2, we find the constrained local maxima and minima of f by introducing two Lagrange multipliers λ and μ. That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0.$$

Example. Page 837, number 40.