Chapter 15. Multiple Integrals

15.2. Double Integrals over General Regions

Note. Let R be a non-rectangular region in the plane. A partition of R is formed in a manner similar to rectangular regions, but we now only take rectangles which lie entirely inside region R (see Figure 15.8 below). As before, we number the rectangles and let ΔA_k be the area of the kth rectangle. Choose a point (x_k, y_k) in the kth rectangle and compute a Riemann sum as

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Again, we define the *double integral* of $f(x, y)$ over R as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x, y)\, dA.$$

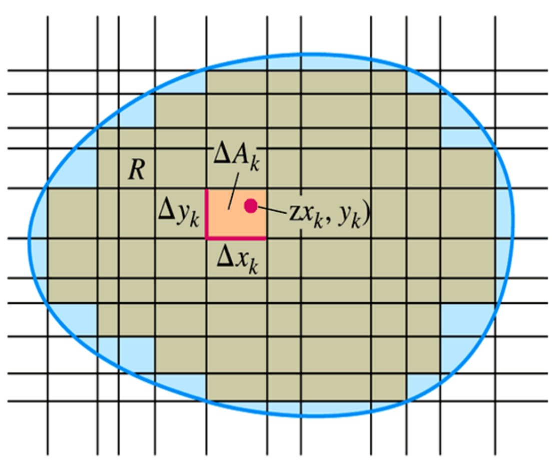


Figure 15.8, Page 859

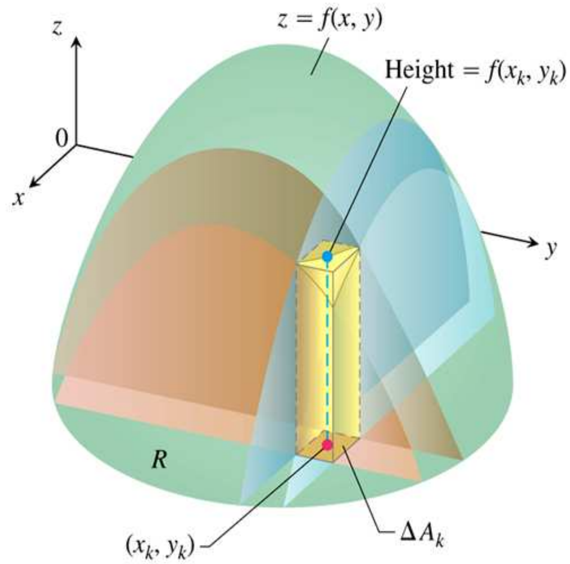


Figure 15.9, Page 860

Definition. When $f(x,y)$ is a positive function over a region R in the xy-plane, we define the *volume* bounded below by R and above by the surface $z = f(x,y)$ to be the double integral of f over R.

Theorem 2. Fubini's Theorem (Stronger Form)

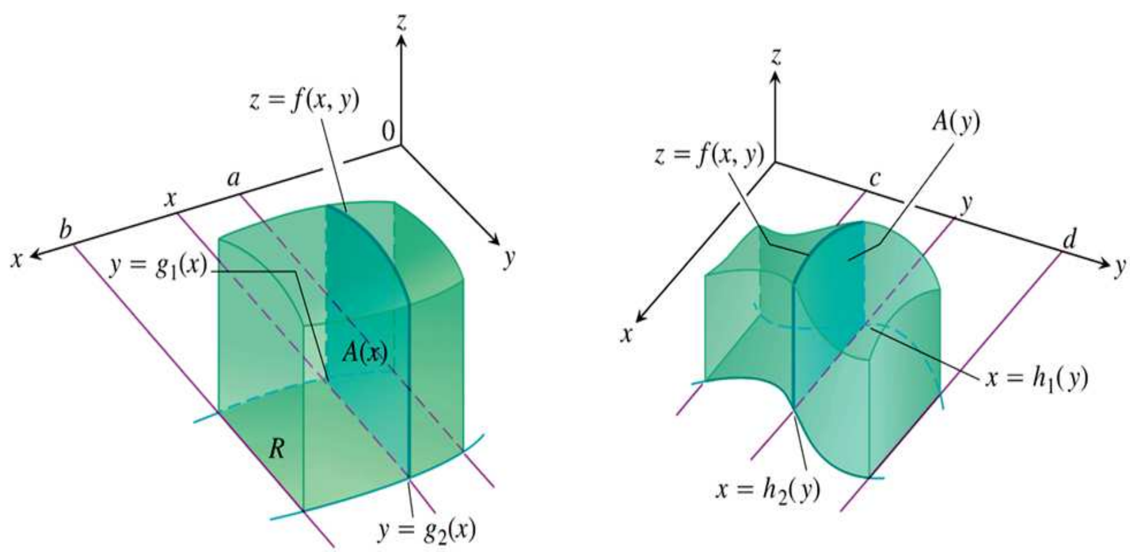
Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $x \in [a,b]$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

2. If R is defined by $y \in [c,d]$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$



Figures 15.10 and 15.11, Pages 860 and 861

Example. Page 866, number 20.

Note. Using Vertical Cross-Sections.

When faced with evaluating $\iint_R f(x,y)\,dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves.

2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).

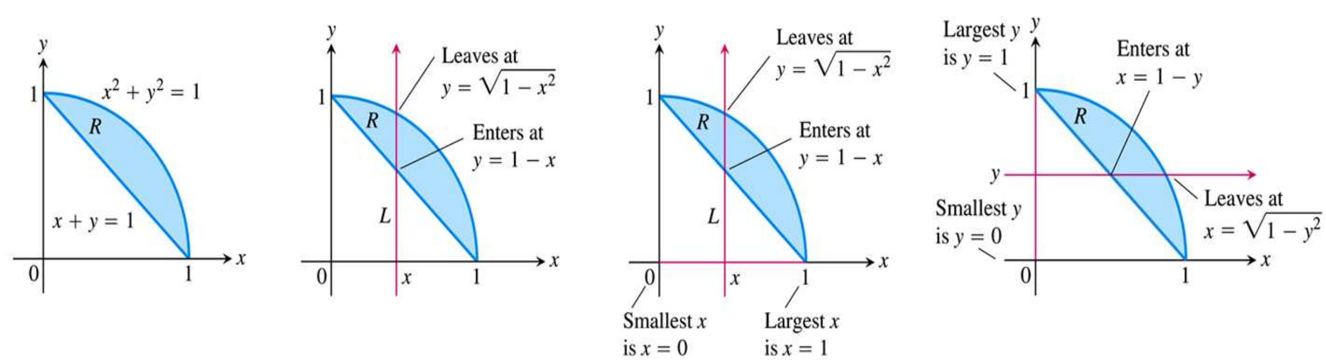
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral shown below is

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

Using Horizontal Cross-Sections.

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral below is

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=1}\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$



Figures 15.14 and 15.15, Page 863

Examples. Page 866, numbers 40 and 50.

Theorem. Properties of Double Integrals.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA$ for any constant c

2. *Sum and Difference:* $\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_A f(x, y)\,dA \pm \iint_R g(x, y)\,dA$

3. *Domination:*

(a) $\iint_R f(x, y)\,dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$ if R is the union of two non-overlapping regions R_1 and R_2

Examples. Page 866, number 58 and Page 867, number 76.

Revised: 1/20/2022