Chapter 15. Multiple Integrals

15.5. Triple Integrals in Rectangular Form

Note. If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes. We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum $S_n = \displaystyle\sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k$. We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k and the norm of the partition $\|P\| = \max\{\Delta x_k, \Delta y_k, \Delta z_k\}$ approaches zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is *integrable* over D. If F is continuous on D and D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. As $\|P\| \to 0$, if the sums S_n approach a limits, then the limit is the *triple integral of F over D*, denoted

$$\lim_{\|P\|\to 0} S_n = \int\int\int_D F(x, y, z)\,dx\,dy\,dz.$$

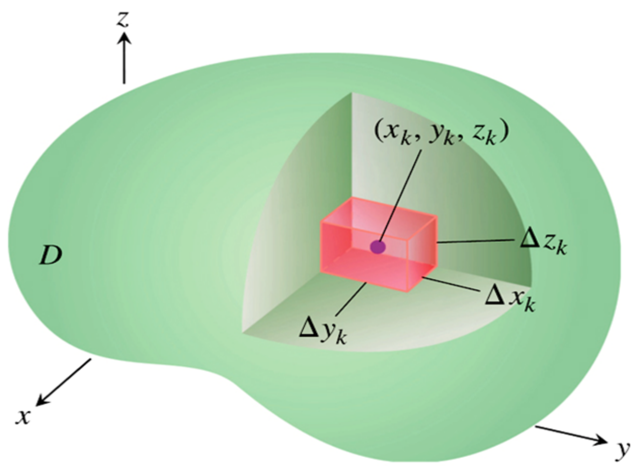


Figure 15.29, Page 877

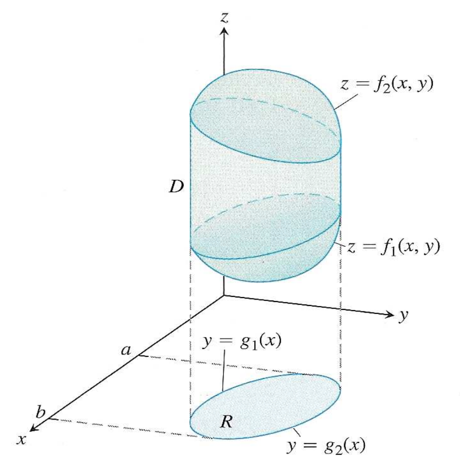
Definition. The *volume* of a closed and bounded region D in space is the triple integral of the function $F(x, y, z) = 1$ over D:

$$V = \int\int\int_D dV.$$

Note. Finding Limits of Integration in the Order $dz\ dy\ dx$

To evaluate $\int\int\int_D F(x, y, z)\, dV$, we illustrate how to find bounds for integrating first with repsect to z, then y, and then x.

1. *Sketch.* Sketch the region D along with its 'shadow' R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



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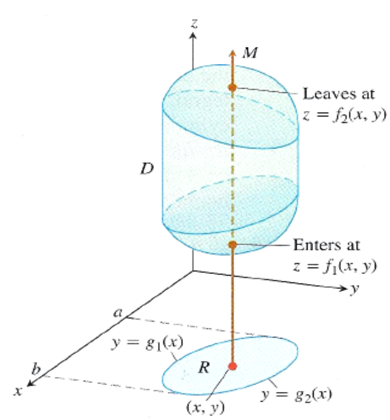
2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



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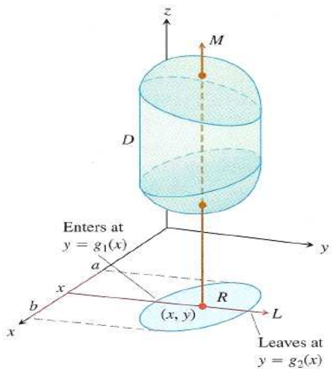
3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



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4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis. These are the x-limits of integration.

In conclusion, the integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Of course, we can modify the order of integration by interchanging the variables. See page 879 for pictures of this.

Example. Example 1, page 880 (set up the integral).

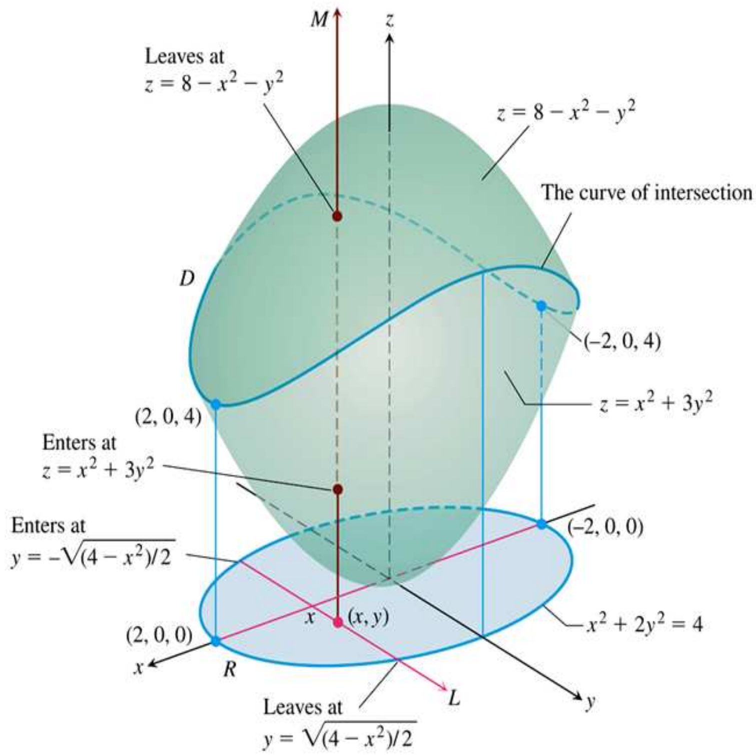

Figure 15.30, Page 880

Examples. Page 883, number 4; page 884, number 26; page 885, numbers 38 and 42.

Definition. The average value of a function F over a region D in space is

$$\text{Average value } = \frac{1}{\text{volume of } D} \int\int\int_D F \, dV.$$