Chapter 15. Multiple Integrals

15.6. Moments and Centers of Mass

Note. If $\delta(x, y, z)$ is the density of an object occupying a region D in space, the integral of δ over D gives the *mass* of the object. The *first moment* of a solid region D about a coordinate plane is defined as the triple integral over D of the distance from a point (x, y, z) in D to the plane multiplied by the density of the solid at that point. For instance, the first moment about the yz-plane is the integral

$$M_{yz} = \int\int\int_D x\delta(x, y, z)\, dV.$$

The *center of mass* is found from the first moments. For instance, the x-coordinate of the center of mass is $\overline{x} = M_{yz}/M$. To summarize, we have:

THREE-DIMENSIONAL SOLID

Mass: $M = \displaystyle\int\int\int_D \delta\, dV$

First moments about the coordinate planes:

$$M_{yz} = \int\int\int_D x\,\delta\, dV,\ M_{xz} = \int\int\int_D y\,\delta\, dV,\ M_{xy} = \int\int\int_D z\,\delta\, dV$$

Center of mass:

$$\overline{x} = \frac{M_{yz}}{M}, \ \overline{y} = \frac{M_{xz}}{M}, \ \overline{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \int\int_R \delta \, dA$

First moments: $M_y = \int\int_R x \, \delta \, dA, \ M_x = \int\int_R y \, \delta \, dA$

Center of mass: $\overline{x} = \dfrac{M_y}{M}, \ \overline{y} = \dfrac{M_x}{M}.$

Note. When the density of a solid object or plate is constant, the center of mass is called the *centroid* of the object. To find a centroid, we set δ equal to 1 and proceed to find $\overline{x}$, $\overline{y}$, and $\overline{z}$ by dividing first moments by masses.

Examples. Page 891, numbers 4 and 16a.

Note. A object's moments tell us about balance and about the torque the object experiences about different axes in a gravitational field. If the object is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy is generated by a shaft rotating at a particular angular velocity. This is where the second moment or moment of inertia comes in.

Note. Think of partitioning the shaft mentioned above into small blocks of mass Δm_k and let r_k denote the distance from the kth block's center of mass to the axis of rotation. If the shaft rotates at a constant angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of mass will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}[r_k\theta] = r_k\frac{d\theta}{dt} = r_k\omega.$$

The block's kinetic energy will be approximately

$$\frac{1}{2}\Delta m_k v_k^2 = \frac{1}{2}\Delta m_k (r_k\omega)^2 = \frac{1}{2}\omega^2 r_k^2 \Delta m_k.$$

The kinetic energy of the shaft will be approximately

$$\sum \frac{1}{2}\omega^2 r_k^2 \Delta m_k.$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$\text{KE}_{\text{shaft}} = \int \frac{1}{2}\omega^2 r^2\,dm = \frac{1}{2}\omega^2 \int r^2\,dm.$$

The factor

$$I = \int r^2\,dm$$

is the *moment of inertia* of the shaft about its axis of rotation, and we see from the above equation that the shaft's kinetic energy is

$$\text{KE}_{\text{shaft}} = \frac{1}{2}I\omega^2.$$

Note. The shaft's moment of inertia is analogous to a linearly moving object's mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia depends on the mass of the shaft and on its distribution of mass. Mass that is farther away from the axis of rotation contributes more to the moment of inertia.

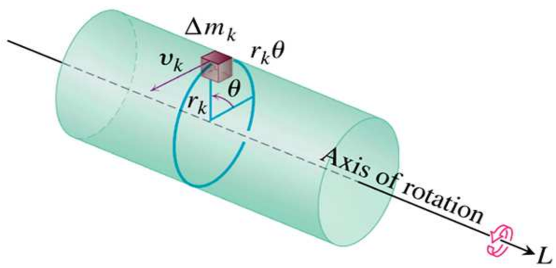

Figure 15.37, Page 888

Note. If $r(x,y,z)$ is the distance from the point (x,y,z) in D to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k,y_k,z_k)\Delta V_k$ about the line L is approximately $\Delta I_k = r^2(x_k,y_k,z_k)\Delta m_k$. The *moment of inertia about* L of the entire object is

$$I_L = \lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta I_k = \lim_{\|P\|\to 0}\sum_{k=1}^{n} r^2(x_k,y_k,z_k)\delta(x_k,y_kz_k)\Delta V_k = \int\int\int_D r^2\delta\,dV.$$

If L is the x-axis, then $r^2 = y^2+z^2$ and

$$I_x = \int\int\int_D (y^2+z^2)\delta(x,y,z)\,dV.$$

Similarly, if L is the y-axis or z-axis we have

$$I_y = \int\int\int_D (x^2+z^2)\delta(x,y,z)\,dV \text{ and } I_z = \int\int\int_D (x^2+y^2)\delta(x,y,z)\,dV.$$

Examples. Page 892, number 22; page 893, number 32.

Note. We also have and define the following moments:

For Two-Dimensional Plates:

About the x-axis: $I_x = \displaystyle\int\int_R y^2 \delta \, dA$

About the y-axis: $I_y = \displaystyle\int\int_R x^2 \delta \, dA$

About the line L: $I_L = \displaystyle\int\int_R r^2(x, y) \delta \, dA$

About the origin (polar moment): $I_0 = \displaystyle\int\int_R (x^2 + y^2) \delta \, dA = I_x + I_y$.

Example. Page 891, number 16.

Note. The stiffness of the beam is a constant times I, the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of I, the stiffer the beam and the less it will bend under a given load. That is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to increase the value of I.

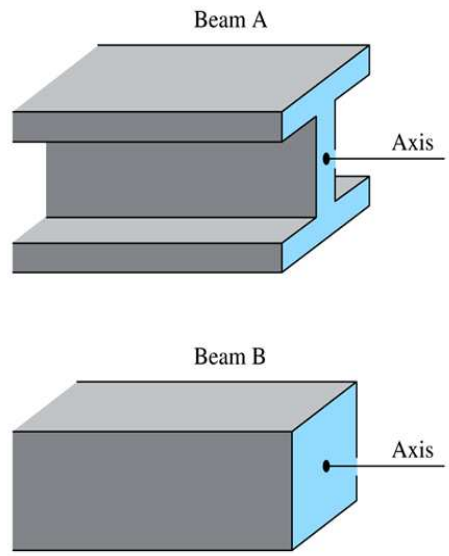

Figure 15.41, Page 891