Chapter 15. Multiple Integrals

15.7. Triple Integrals in Cylindrical and Spherical Coordinates

Definition. *Cylindrical coordinates* represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane

2. z is the rectangular vertical coordinate.

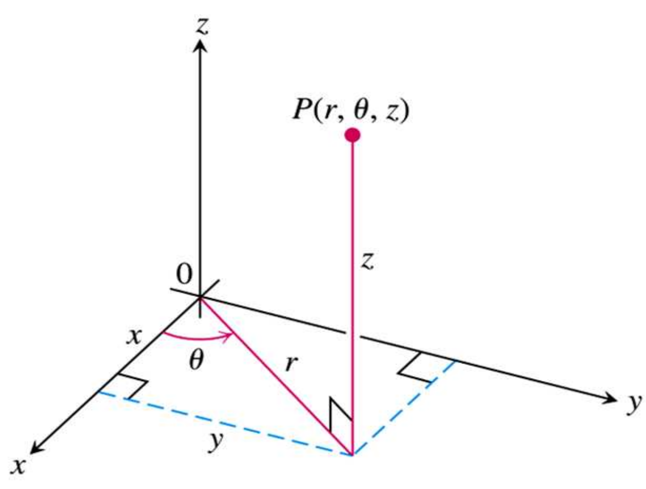


Figure 15.42, Page 893

Note. The equations relating rectangular (x, y, z) and cylindrical (t, θ, z) coordinates are

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

$$r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

Note. In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis. The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

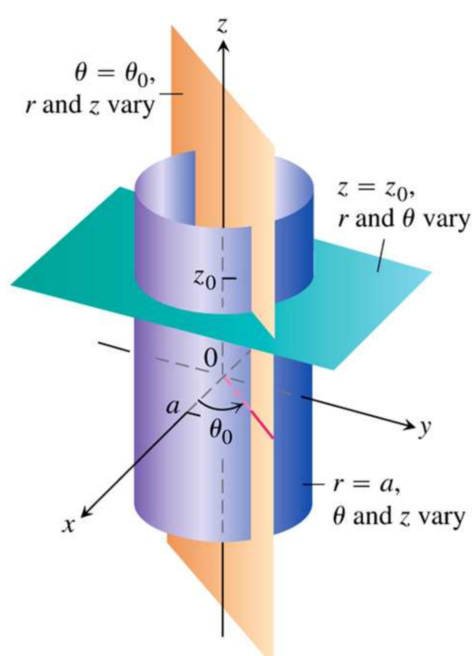


Figure 15.43, Page 894

Note. When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the kth cylindrical wedge, r, θ and z change by Δr_k, $\Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the *norm* of the partition. We define the triple integral as a limit of Riemann sums using these wedges. Thee volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz. For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k \Delta r_k \Delta\theta_k$. So $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{\|P\|\to 0} S_n = \int\int\int_D f\, dV = \int\int\int_D f\, dz\, r\, dr\, d\theta.$$

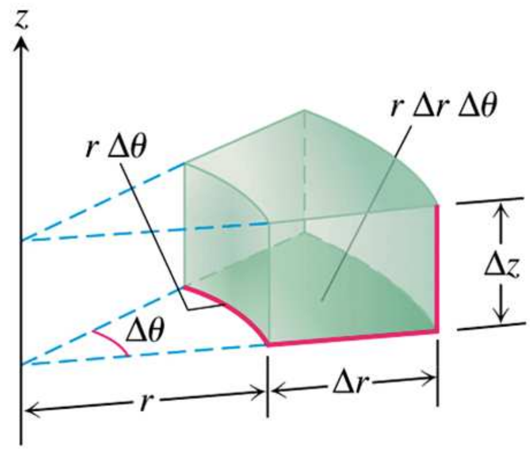


Figure 15.44, Page 894

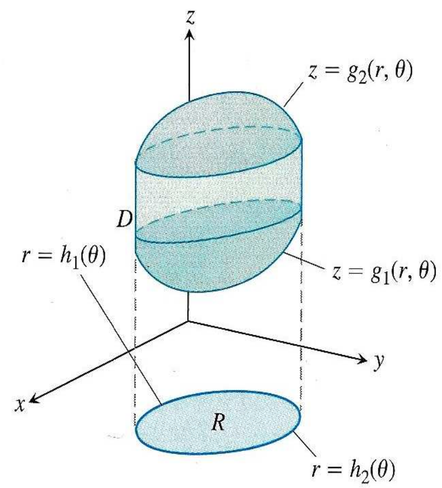
Example. Page 901, number 4.

How to Integrate in Cylindrical Coordinates

To evaluate $\displaystyle\int\int\int_D f(r,\theta,z)\,dV$ over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-

plane. Label the surfaces and curves that bound D and R.



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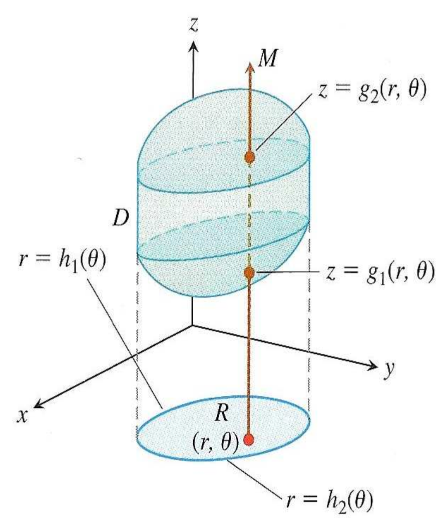
2. *Find the z-limits of integration.* Draw a line M passing through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



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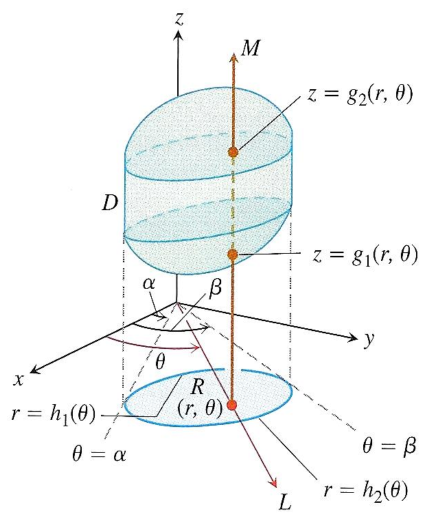
3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



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4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits if integration. The integral is

$$\int\int\int_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example. Page 902, number 18.

Definition. *Spherical coordinates* represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin (notice that $\rho > 0$).

2. ϕ is the angle $\vec{OP}$ makes with the positive z-axis ($\phi \in [0, \pi]$).

3. θ is the angle from cylindrical coordinate ($\theta \in [0, 2\pi]$).

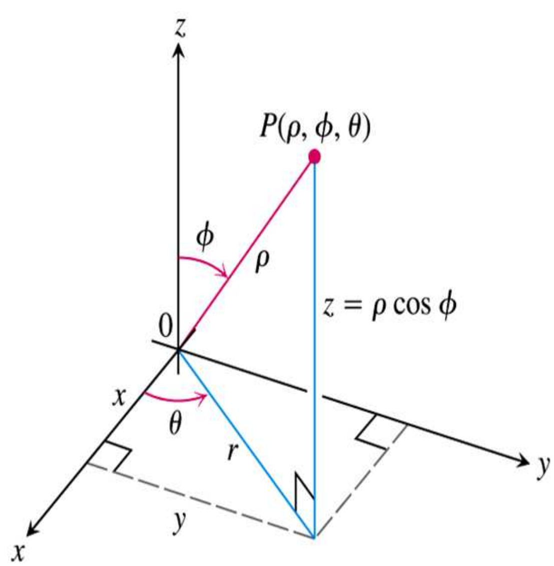


Figure 15.47, Page 897

Note. The equation $\rho = a$ describes the sphere of radius a centered at the origin. The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis.

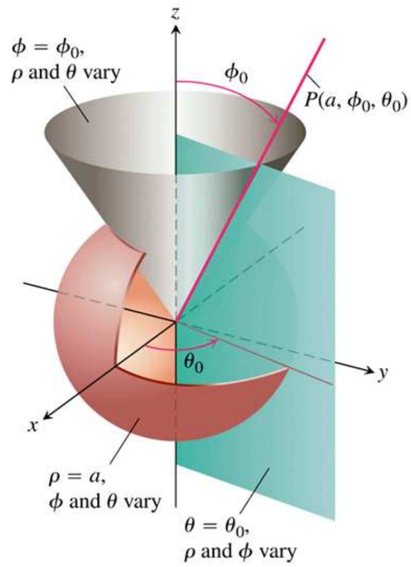


Figure 15.48, Page 897

Note. The equations relating spherical coordinates to Cartesian coordinates and cylindrical coordinates are

$$r = \rho \sin\theta,\ \ x = r\cos\theta = \rho\sin\phi cos\theta,$$

$$z = \rho\cos\phi,\ \ y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Note. When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given be the changes $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ in ρ, θ, and ϕ. Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ are all small. It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge. The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k.$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when f is continuous:

$$\lim_{\|P\| \to 0} S_n = \int\int\int_D f(\rho, \phi, \theta) \, dV = \int\int\int_D f(\rho, \phi, \theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

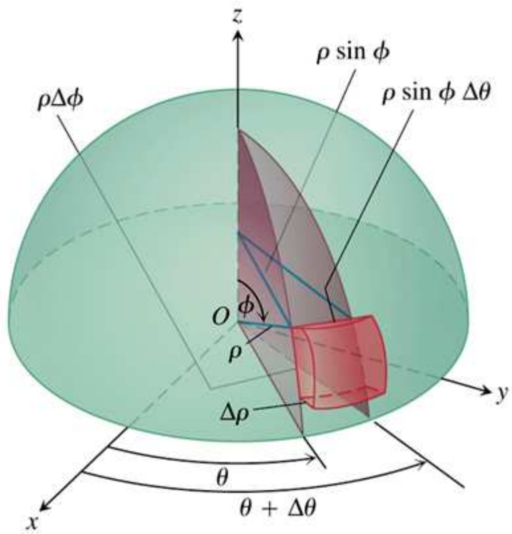


Figure 15.51, Page 898

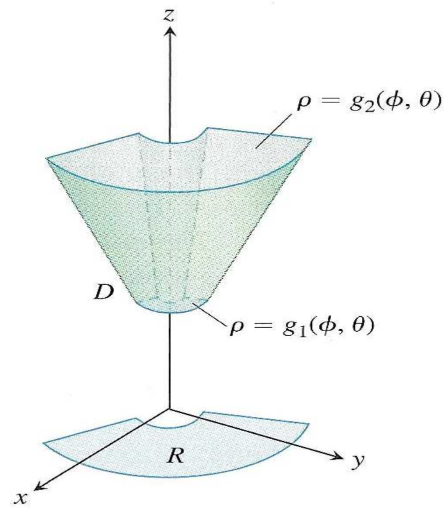
Example. Page 902, number 26.

How to Integrate in Spherical Coordinates

To evaluate $\displaystyle\int\int\int_D f(\rho,\phi,\theta)\,dV$ over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-

plane. Label the surfaces and curves that bound D and R.



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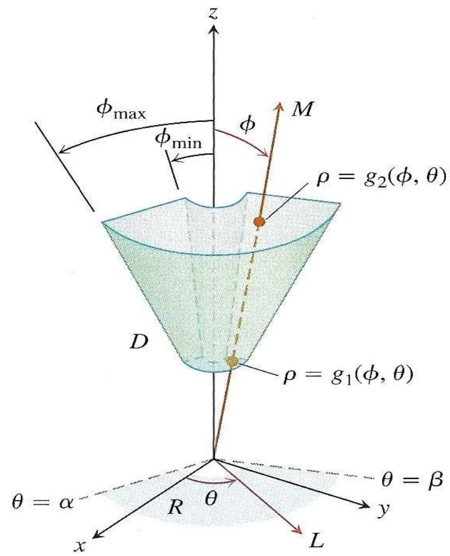
2. *Find the ρ-limits of integration.* Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



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3. *Find the ϕ-limits of integration.* For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ-limits of integration.

4. *Find the θ-limits of integration.* The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\int\int\int_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\; d\theta.$$

Example. Page 903, number 34.

Note. In summary, we have the following relationships.

Cylindrical to Rectangular	**Spherical to Rectangular**	**Spherical to Cylindrical**
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\theta$	$\theta = \theta$

In terms of the differential of volume, we have

$$dV = dx\,dy\,dz = dz\,r\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Examples. Page 903, number 46. Page 904, number 54.