Chapter 15. Multiple Integrals

15.8. Substitutions in Multiple Integrals

Note. Suppose that a region G in the uv-plane is transformed one-to-one into the region R n the xy-plane by equations of the form

$$x = g(u, v),\ y = h(u, v).$$

We call R the *image* of G under the transformation, and G the *preimage* of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is hte integral of $f(x,)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G? The answer is: If g, h, and f have continuous partials derivatives and $J(u, v)$ is zero only at isolated points, then

$$\int\int_R f(x, y)\, dx\, dy = \int\int_G f(g(u, v), h(u, v))|J(u, v)|\, du\, dv.$$

The factor $J(u, v)$, whose absolute value appears above, is the *Jacobian* of the coordinate transformation. It measures how much the transformation is expanding or contracting the area around a point in G as G is transformed into R.

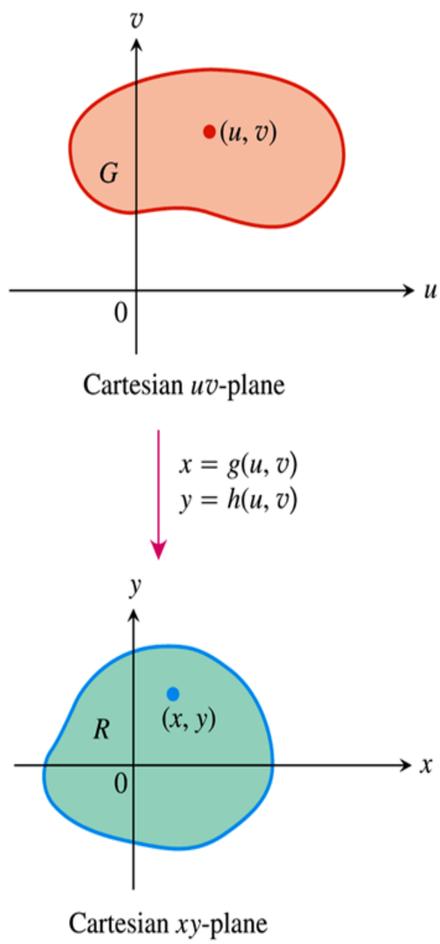

Figure 15.53, Page 905

Definition. The *Jacobian determinant* or *Jacobian* of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Note. The proof of the above is "intricate and properly belongs to a course in advances calculus. We do not give the derivation here."

Note. See Example 1, page 905, for use of the Jacobian to relate integration in rectangular coordinates to integrals in polar coordinates (as before).

Example. Page 912, number 2.

Example. Example 3, page 907. Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\,dy\,dx$.

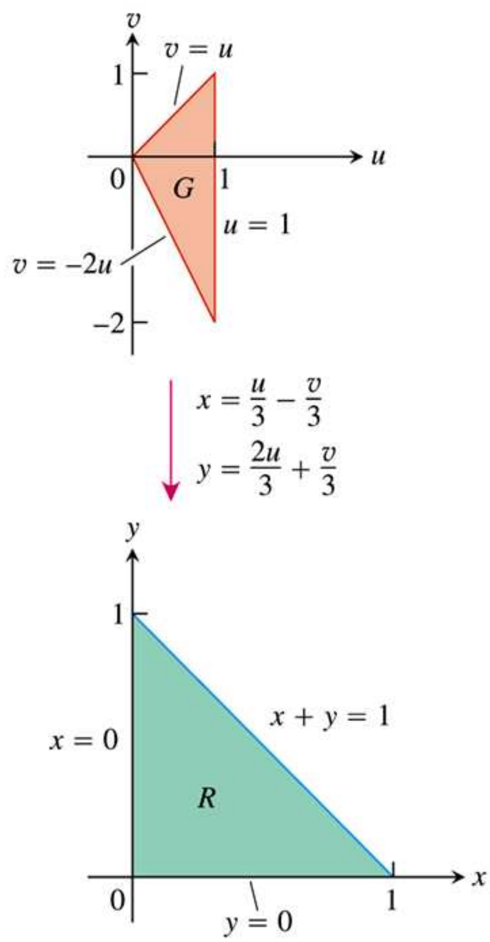

Figure 15.56, Page 907

Note. Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u,v,w),\ \ y = h(u,v,w),\ \ z = k(u,v,w).$$

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related o the integral of $H(u, v, w)$ over G by the equation

$$\int\int\int_D F(x, y, z)\, dx\, dy\, dz = \int\int\int_G H(u, v, w)|J(u, v, w)|\, du\, dv\, dw.$$

The factor $J(u, v, w)$ whose absolute value appears in this equation, is the *Jacobian determinant*

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates.

Example. Page 913, number 18.

Note. See page 910 for computations showing the Jacobian determinate relating rectangular coordinates to spherical coordinates and cylindrical coordinates is consistent with the results of section 15.7.

Example. Page 913, number 22.