6.5 Quadratic Forms: Orthogonal Diagonalization

Definition: A matrix A is *symmetric* if and only if $A = A^T$.

Theorem 1 *Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix A are orthogonal.*

Proof: Let X and Y be two eigenvectors for A that correspond to eigenvalues λ and β where $\lambda \neq \beta$. Note that

$$AX \cdot Y = (AX)^T Y = X^T A^T Y = X^T (AY) = X \cdot AY.$$

It follows that

$$\begin{aligned} (\lambda X) \cdot Y &= (X \cdot \beta Y) \\ \Rightarrow \qquad 0 &= (\lambda - \beta)(X \cdot Y) \end{aligned}$$

Since $\lambda \neq \beta$, this implies $X \cdot Y = 0$, proving the orthogonality. Q.E.D.

Spectral Theorem *Let A be an $n \times n$, real symmetric matrix. Then A has an orthonormal diagonalizing basis. Hence, there is an orthogonal matrix P such that $P^T AP$ is diagonal (or $A = PDP^{-1}$ — $P^T = p^{-1}$ since P is orthogonal).*

Definition: For matrix $A = [a_{ij}]$ and vector $X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$, the solution set to

$$\sum a_{ij} x_i x_j = d$$

is called a *quadratic variety*. We may write this equation as

$$X^T AX = d.$$

The function which transforms X into $X^T AX$ is called a *quadratic form* and the matrix A is called the *matrix for the form*.

Note: Given any quadratic form, we may always find a symmetric matrix A which represents it.

Definition: A quadratic form is said to be in *standard form* if its matrix is diagonal. In this case we have

$$a_{11} x_1^2 + a_{22} x_2^2 + \cdots + a_{nn} x_n^2 = d.$$

Principal Axis Theorem *Let a quadratic variety be defined by the equation*

$$d = X^T AX$$

where $X = [x_1, x_2, \ldots, x_n]^T$*,* A *is an* $n \times n$ *symmetric matrix, and* d *is a scalar. Then the corresponding quadritic form is in standard form relative to the coordinates defined by any orthonormal diagonalizing basis for* A*. Furthermore, in these coordinates, the variety is given by*

$$\lambda_1(x_1')^2 + \lambda_2(x_2')^2 + \cdots + \lambda_n(x_n')^2 = d$$

where the λ_i *are the eigenvalues of* A *(listed in an order consistent with the ordering of the basis).*

Theorem 2 *Let* A *be a real, symmetric,* $n \times n$ *matrix. Then every eigenvalue of* A *is real. In particular,* A *has al least one real eigenvector.*

Proof: Suppose that X is a complex eigenvector for A corresponding to the eigenvalue λ. Then, $\overline{X}$ is also an eigenvector for A corresponding to $\overline{\lambda}$ is not real so that $\lambda \neq \overline{\lambda}$.

In Theorem 1, we proved that for a symmetric matrix, if X and Y are eigenvectors corresponding to different eigenvalues, then $X \cdot Y = 0$. An inspection of the proof of Theorem 1 shows that the proof is valid for