Section 1.8. The Growth of Functions

Note. In this section we give several rate of growth definitions and illustrate them.

Definition 1.8.1. Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. Then $f(x)$ is $O(g(x))$ (read"$f(x)$ is of *big-oh* of $g(x)$") if there are constants C and k such that $|f(x)| \leq C|g(x)|$ for all $x > k$.

Example. Page 90 Number 2a.

Definition. If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$ then f and g are of the *same order*.

Theorem 1.8.1. Let $f(x) = z_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $f(x)$ is $O(x^n)$.

Proof. If $x > 1$ then

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^n + \cdots + |a_1|x^n + |a_0 x^n \\
&= x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|).
\end{aligned}$$

So for $x > 1$, $|f(x)| \leq Cx^n$ where $C = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$. ∎

Theorem 1.8.2. Suppose $f_1(x)$ is $O(g(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1+f_2)(x)$ is $O(\max(g_1(x), g_2(x)))$.

Proof. We hypothesize $|f_1(x)| \leq C_1|g(x)|$ for $x > k$, and $|f_2(x)| \leq C_2|g_2(x)|$ for $x > k_2$. Then

$$\begin{aligned} |(f_1+f_2)(x)| &\leq |f_1(x)| + |f_2(x)| \\ &\leq C_1|g_1(x)| + C_2|g_2(x)| \text{ for } x > \max(k_1, k_2) \\ &\leq (C_1 + C_2)|g(x)| \text{ where } g(x) = \max(g_1(x), g_2(x)) \\ &= C|g(x)| \text{ where } C = \max(C_1, C_2). \end{aligned}$$

So $f_1 + f_2)(x)$ is $O(\max(g_1(x), g_2(x))$, as claimed. ∎

Example. Page 90 Number 81, 8b, 8c.

Theorem 1.8.3. Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$.

Proof. Suppose $|f_1(x)| \leq C_1g_1(x)$ for $x > k_1$ and $|f_2(x)| \leq C_2g_2(x)$ for $x > k_2$. Then

$$\begin{aligned} |(f_1f_2)(x)| &= |f_1(x)||f_2(x)| \\ &\leq C_1|g_1(x)|C_2|g_2(x)| \text{ for } x > \max(k_1, k_2) \\ &= C|(g_1g_2)(x)| \text{ where } C = C_1C_2. \end{aligned}$$

∎

Example. Page 85 Example 5. Show $\log n!$ is $O(n \log n)$.

Definition 1.8.2. Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. Then $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that $|f(x)| \geq C|g(x)|$ for $x > k$. We say "$f(x)$ is *big-Omega* of $g(x)$."

Example. Page 91 Number 30.

Definition 1.8.3. Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$. We say "$f(x)$ is *bog-Theta* of $g(x)$" or "$f(x)$ is of *order* $g(x)$."

Example. Page 91 Number 24a and 24e.

Theorem 1.8.4. Let $f(x) = a_x x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$. Then $f(x)$ is of order x^n.

Example. Page 91 Number 38 and Page 92 Number 51b.

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