Section 2.6. Matrices

Note. In this section we review some properties of matrices from Linear Algebra and define the "Boolean product" of two matrices with entries of 0 and 1.

Definition 2.6.1. A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is an $m \times n$ *matrix*. Two matrices are *equal* if they are the same size and corresponding entries are equal.

Definition 2.6.2. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The ith row of A is the $1 \times n$ matrix $[a_{i1}\ a_{i2}\ \cdots\ a_{in}]$. The jth column of A is the $m \times 1$ matrix $\begin{bmatrix} 1_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$. The (i, j)th element or *entry* of A is the element a_{ij}. Sometimes we denote $A = [a_{ij}]$.

Definition 2.6.3. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The *sum* of A and B, denoted $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th entry.

Example. Page 159 Number 2a.

Definition 2.6.4. Let A be an $m \times k$ matrix and B a $k \times n$ matrix. The *product* of A and B, denoted AB, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{y=1}^{k} a_{it}b_{tj}.$$

Example. Page 159 Number 4a. Notice that we do not in general have commutivity of matrix multiplication. That is, there are A and B of appropriate sizes where $AB \neq BA$.

Note. The text gives an algorithm to calculate the product of two matrices. If the matrices are square of size $n \times n$ then it requires n^3 multiplications and $n^2(n-1)$ additions (and so is $O(n^3)$).

Definition 2.6.5. The *identity matrix of order* n is the $n \times n$ matrix $I_n = [\delta_{ij}]$ where

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j. \end{cases}$$

Note. If A is $m \times n$ then $A\mathcal{I}_n = \mathcal{I}_m = A$.

Definition 2.6.6. Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of A, denoted A^T, is an $n \times m$ matrix obtained by interchanging the rows and columns of A. That is, $A^T = [b_{ij}]$ where $b_{ij} = a_{ji}$.

Example. Page 160 Number 16.

Definition 2.6.7. A matrix A is *symmetric* if $A = A^T$.

Definition. In a square matrix, the elements a_{ii} form the *diagonal* of A.

Definition 2.6.8. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices all of whose entries are either 0 or 1. The *join* of A and B, denoted $A \vee B$, is the matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. Recall

$$a_{ij} \vee b_{ij} = \begin{cases} 1 \text{ if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 \text{ otherwise.} \end{cases}$$

The *meet* of A and B, denoted $A \wedge B$, is the matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. Recall

$$a_{ij} \wedge b_{ij} = \begin{cases} 1 \text{ if } a_{ij} = b_{ij} = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Example. Page 160 Number 28.

Definition 2.6.9. Let $A = [a_{ij}]$ be an $m \times k$ "zero-one" matrix and $B = [b_{ij}]$ be a $k \times n$ "zero-one" matrix. Then the *Boolean product* of A and B, denoted $A \odot B$ is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \cdots \vee (a_{ij} \vee b_{kj}).$$

Examples. Page 161 Numbers 30 and 36.

Definition 2.6.10. Let A be a square zero-one matrix and let R be a positive integer. The rth *Boolean power* of A, denoted $A^{[r]}$, is

$$A^{[r]} = \underbrace{A \odot A \odot A \odot \cdots \odot A}_{r \text{ times}}.$$

Note. The Boolean powers of A will be used in our study of paths in graphs. The text gives an algorithm for calculating Boolean products of graphs on page 158.

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