Chapter 3. Mathematical Reasoning

Section 3.1. Method of Proof

Note. In this section we give several logical arguments that are commonly used in mathematical proofs.

Note. The book makes the following definitions:

1. A *theorem* is a statement that can be shown to be true.

2. The underlying assumptions about "mathematical structures" are called *axioms* or *postulates*.

3. A *lemma* is a simple theorem used in the proof of other theorems.

4. A *corollary* is a proposition that can be established directly (i.e., easily) form a theorem.

5. A *conjecture* is a statement whose truth value is unknown.

Note. We operate under the following "Rules of Inference." We represent a Rule of Inference as a hypothesis over a conclusion.

Rule of Inference	**Name**
$\dfrac{p}{\therefore p \vee q}$	Addition
$\dfrac{p \wedge q}{\therefore p}$	Simplification
$\dfrac{q}{\therefore p \wedge q}$	Conjunction
$\dfrac{p \to q}{\therefore q}$	*Modus Ponens* (Law of Detachment)
$\dfrac{\begin{array}{c}\neg q \\ p \to q\end{array}}{\therefore \neg p}$	*Modus Tollens* (Contrapositive)
$\dfrac{\begin{array}{c}p \to q \\ q \to r\end{array}}{\therefore r}$	Hypothetical Syllogism (Transitivity of Implication)
$\dfrac{\begin{array}{c}p \vee q \\ \neg p\end{array}}{\therefore q}$	Disjunction Syllogism

Examples. Page 183 Numbers 2 and 8a.

Note. We have the Rules of Inference for Quantified Statements:

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(c) \text{ if } c \in U}$	Universal Instantiation
$\dfrac{P(c) \text{ for arbitrary } c \in U}{\therefore \forall x P(x)}$	Universal Generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some } c \in U}$	Existential Instantiation
$\dfrac{P(c) \text{ for some } c \in U}{\therefore \exists x P(x)}$	Existential Generalization

Example. Page 174 Example 13.

Note. The text mentions several types of proofs of $p \to q$:

1. In a *direct proof* we assume p and demonstrate q.

2. In an *indirect proof* we assume $\neq q$ and show $\neg p$ (i.e., we give a direct proof of the contrapositive $\neg q \to \neg p$).

3. A *vacuous proof* can be given if p can be shown to be false.

4. A *Trivial proof* can be given if a can be shown to be T regardless of the truth value of p.

5. A *proof be contradiction* assume as hypothesis $\neg p$ and show $\neg p \to q$ where q is a contradiction (i.e., a is always F).

Example. A proof of $\exists x P(x)$ is called an *existence proof*. If an element a if found such that $P(a)$ then the proof is *constructive*. Otherwise it is nonconstructive.

Example. Prove that $f(x) = 7x^5 + 4x^4 - 3x^3 + 2x^2 - \pi x - 1$ has a solution in $[0, 1]$.

Solution. We observe that $f(0) = -1$ and $f(1) = 9 - \pi > 0$. Since f is a polynomial, then it is continuous. So by the Intermediate Value Theorem, there exists $y \in [0, 1]$ such that $f(y) = 0$. This is a nonconstructive proof since the existence of the solution y is found without explicitly finding the value of y.

Note. To prove $\neg(\forall x P(x))$ we need only show $\exists x \neg P(x)$. Such a proof is by *counterexample*.

Example. Page 184 Number 62. See also Page 176 Example 18.

Revised: 4/4/2019