Section 4.2. Pigeonhole Principle

Note. In this section we state another counting principle.

Theorem 4.2.1. The Pigeonhole Principle.
If $k+1$ or more objects are placed into k boxes then there is at least one box containing two or more of the objects.

Example. Page 245 Example 1. Among any group of 367 people, at least two people must have the same birthday (since there are 366 possible birthdays).

Theorem 4.2.2. The Generalized Pigeonhole Principle.
If n objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects.

Examples. Page 249 Numbers 14 and 34.

Definition. A *subsequence* of sequence $a_1, a_2, \ldots$ is a sequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ where $1 \leq i_1 \leq i_2 < i_3 < \cdots$. A sequence is *strictly increasing* if each term is larger than the one that precedes it A sequence is *strictly decreasing* if each term is smaller than the one that precedes it.

Theorem. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing, or strictly decreasing.

Proof. Let $a_1, a_2, \ldots, a_{n^2+1}$ be the distinct numbers in the sequence. With term a_k associate the pair (i_k, d_k) where i_k is the length of the longest increasing subsequence *starting* at a_k, and d_k is the length of the longest decreasing sequence starting at a_k.

ASSUME there are no decreasing or increasing subsequences of length $n+1$ (we give a proof by contradiction). Then i_k and d_k are both positive integers $\leq n$, for $k = 1, 2, \ldots, n^2 + 1$. So there are (by the Product Rule) n^2 possible ordered pairs (i_k, d_k). By the Pigeonhole Principle, two of these pairs must be equal. That is, for some a_s and a_t with $s < t$ we have $i_s = i_t$ and $d_s = d_t$.

If $a_s < a_t$, then since $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s and following it with the increasing subsequence of length i_t which begins at a_t. Therefore $i_s = i_t + 1 \neq i_t$, a CONTRADICTION.

If $a_x > a_t$, we can similarly conclude that $d_s = d_t + 1 + d_t$, a CONTRADICTION.

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