Section 4.3. Permutations and Combinations

Note. In this section we give more counting results.

Definition. A *permutation* of a set of distinct objects is an ordered arrangement of the objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Theorem 4.3.1. The number of r-permutations of a set with n distinct elements is

$$P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)).$$

Note. We have $P(n,r) = \dfrac{n!}{(n-r)!}$.

Example. Page 258 Number 10.

Definition. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

Theorem 4.3.2. The number of r-combinations of a set with n elements, where n is a positive integer and r is an integer with $0 \le r \le n$ is

$$C(n,r) = \frac{n!}{r!(n-r)!}.$$

Example. Page 258 Number 18.

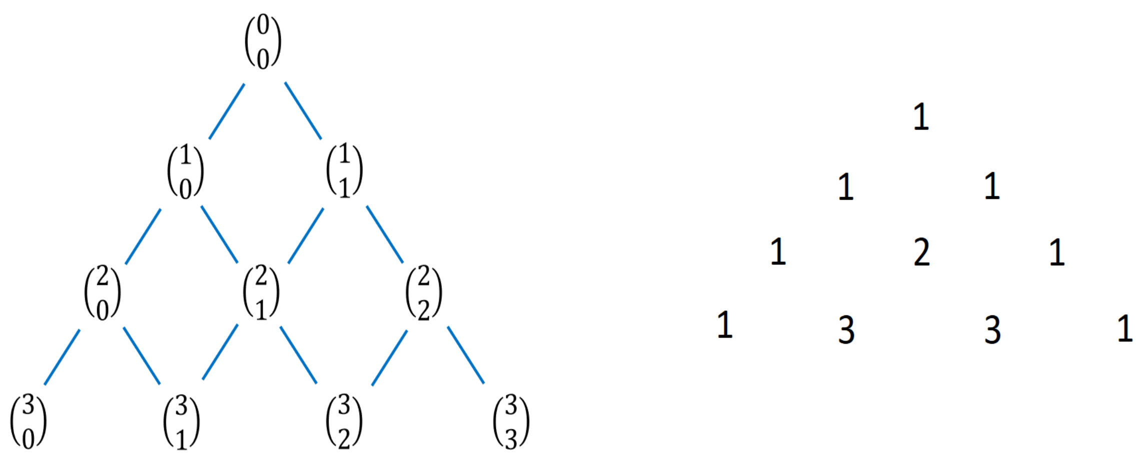
Corollary 4.3.1. Let $n, r \in \mathbb{Z}^+$ with $r \leq n$. Then $C(n,r) = C(n, n-r)$.

Theorem 4.3.3. Pascal's Identity.

Let $n, k \in \mathbb{Z}^+$ with $n \geq k$. Then

$$C(n+1, k) = C(m, k-1) + C(n, k).$$

Note. The above result leads us to Pascal's Triangle



Theorem 4.3.5. Vandermonde's Identity.

Let $m, n, r \in \mathbb{Z} \cup \{0\}$ with $r \leq m$ and $r \leq n$. Then

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k).$$

Theorem 4.3.6. Binomial Theorem.

Let x and y be variables and let $n \in \mathbb{Z}^+$. Then

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j.$$

Example. Use the Binomial Theorem to prove for $n \in \mathbb{Z}^=$: $\dfrac{d}{dx}[x^n] = nx^{n=1}$.

Examples. Page 259 Numbers 36 and 54.

Theorem 4.3.4. Let $n \in \mathbb{Z}^+$. Then $\displaystyle\sum_{k=0}^{n} C(n,k) = 2^n$.

Proof. Let $x = y = 1$ in the Binomial Theorem and the result follows. ∎

Theorem 4.3.7. Let $n \in \mathbb{Z}^+$. Then

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0.$$

Proof. Let $x = 1$ and $y = -1$ in the Binomial Theorem and the result follows. ∎

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