Section 6.5. Equivalence Relations

Note. In this section we define an equivalence relation on a set and show that the equivalence classes of such a relation partition the set on which it is defined (this is done in Theorem 6.5.2). The will play a large role in Introduction to Modern Algebra (MATH 4127/5127).

Definition 6.5.1. A relation on a set A is an *equivalence relation* if it is reflexive ($(a,a) \in R$), symmetric ($(a,b) \in R \Leftrightarrow (b,a) \in R$), and transitive ($(a,b),(b,c) \in R \Rightarrow (a,c) \in R$).

We can represent relations with 01 matrices. Define for relation R matrix $M_R = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Example. Page 409 Example 4. Let $m \in \mathbb{N}$, $m > 1$. Prove that

$$R = \{(a,b) \mid a \equiv b (\text{mod } m)\}$$

is an equivalence relation on $\mathbb{Z}$.

Definition 6.5.2. Let R be an equivalence relation on set A. The set of all elements that are related to $a \in A$ is the *equivalence class of* a, denoted $[a]_R$.

Theorem 6.5.1. Let R be an equivalence relation on set A. The following are equivalent:

1. aRb,

2. $[a] = [b]$, and

3. $[a] \cap [b] \neq \varnothing$.

Definition. A partition of a set S is a collection of disjoint nonempty subsets of S that union to give S.

Theorem 6.5.2. Let R be an equivalence relation on set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition of S there is a relation R with equivalence classes the same as the sets in the partition.

Examples. Page 413 Numbers 12 and 26.

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