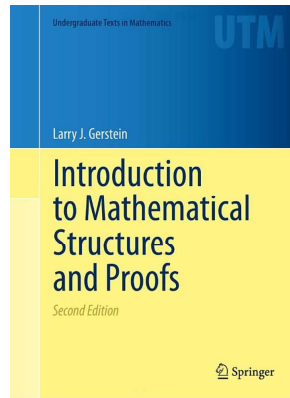


# Mathematical Reasoning

## Chapter 2. Sets

### 2.1. Fundamentals—Proofs of Theorems



## Theorem 2.5

**Theorem 2.5.** There is exactly one empty set. That is, all empty sets are equal.

**Proof.** Let  $\emptyset$  and  $\emptyset'$  be two arbitrary empty sets. Recall that two sets are equal if they have the same members. That is,  $\emptyset = \emptyset'$  means the equivalence  $x \in \emptyset \Leftrightarrow x \in \emptyset'$  for every element  $x$ . But for every  $x$ , the statements  $x \in \emptyset$  and  $x \in \emptyset'$  are both false (since  $\emptyset$  and  $\emptyset'$  have no elements). So the implications  $x \in \emptyset \Rightarrow x \in \emptyset'$  and  $x \in \emptyset' \Rightarrow x \in \emptyset$  are both true. That is,  $x \in \emptyset \Leftrightarrow x \in \emptyset'$  for every element  $x$ , and so  $\emptyset = \emptyset'$ . Since  $\emptyset$  and  $\emptyset'$  are arbitrary empty sets, then the claim holds.  $\square$

**Note.** This proof showed that the desired implications hold “for all  $x$ ,” when there were in fact no such  $x$ . That is, the hypotheses of the implications are false, and hence the implications are true. When a result is proved under such conditions, the result is said to hold *vacuously*.