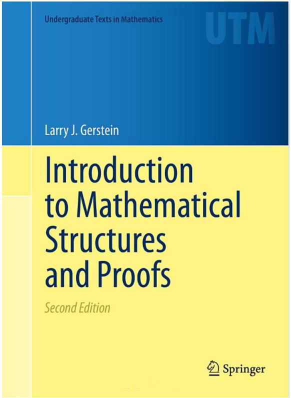
Mathematical Reasoning

Chapter 2. Sets

2.1. Fundamentals—Proofs of Theorems



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Proof. Let $\varnothing$ and $\varnothing'$ be two arbitrary empty sets. Recall that two sets are equal if they have the same members. That is, $\varnothing = \varnothing'$ means the equivalence $x \in \varnothing \Leftrightarrow x \in \varnothing'$ for every element x. But for every x, the statements $x \in \varnothing$ and $x \in \varnothing'$ are both false (since $\varnothing$ and $\varnothing'$ have no elements). So the implications $x \in \varnothing \Rightarrow x \in \varnothing'$ and $x \in \varnothing' \Rightarrow x \in \varnothing$ are both true. That is, $x \in \varnothing \Leftrightarrow x \in \varnothing'$ for every element x, and so $\varnothing = \varnothing'$. Since $\varnothing$ and $\varnothing'$ are arbitrary empty sets, then the claim holds. □

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Note. This proof showed that the desired implications hold "for all x," when there were in fact no such x. That is, the hypotheses of the implications are false, and hence the implications are true. When a result is proved under such conditions, the result is said to hold *vacuously*.

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