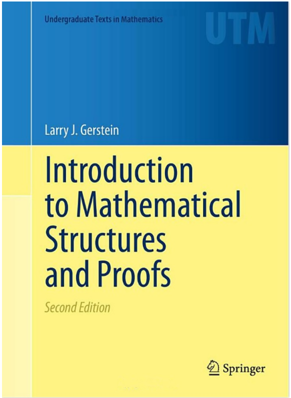
Mathematical Reasoning

Chapter 2. Sets

2.10. Mathematical Induction and Recursion—Proofs of Theorems



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Let n_0 be an integer. Suppose P is a property such that

(a) $P(n_0)$ is true.

(b) For every integer $k \geq n_0$, the following conditional statement is true:

If $P(n)$ is true for every n satisfying $n_0 \leq n \leq k$, then $P(k+1)$ is true.

The $P(n)$ is true for every integer $n \geq n_0$.

Proof. We give a proof by contradiction. ASSUME $P(n)$ is not true for every integer $n \geq n_0$. That is, assume there is some integer $m \geq n_0$ for which $P(m)$ is false. By hypothesis (a) we have $m = n_0 + t$ for some $t \geq 1$. Let t be the least natural number for which $P(n_0 + t)$ is false; such a t exists by the Well-Ordering Principle on the natural numbers.

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Proof (continued). Then every integer n where $n_0 \leq n \leq n_0 + (t-1)$ has property P. But then hypothesis (b) implies that the statement $P((n_0 + (t-1)) + 1) = P(n_0 + t)$ is true, CONTRADICTING the assumption above. So the assumption that $P(n)$ is not true for every integer $n \geq n_0$ is false, and hence $P(n)$ is true for every integer $n \geq n_0$, as claimed. □

Example 2.67

Example 2.67. For every $n \in \mathbb{N}$, $1 + 2 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$.

Proof. First, we establish the basis step: $P(1) = 1 = (1+1)/2$.

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Proof. First, we establish the basis step: $P(1) = 1 = (1+1)/2$. For the induction step, we have

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \text{ by the induction hypothesis } P(k) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Therefore, $P(k+1)$ is true and by the Principle of Mathematical Induction, the equality holds for all $n \in \mathbb{N}$. □

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Theorem 2.69

Theorem 2.69. If S is a set with n elements then the power set $P(S)$ has 2^n elements.

Proof. We use the Principle of Mathematical Induction with $n_0 = 0$. For the basis step, we have $P(\varnothing) = \{\varnothing\}$ so that the power set of $\varnothing$ has $2^{n_0} = 2^0 = 1$ elements.

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Example 2.70

Example 2.70. For every integer $n \geq 0$, the number $4^{2n+1} + 3^{n+2}$ is a multiple of 13.

Proof. We take the proposition $P(n)$ as $4^{2n+1} + 3^{n+2}$ is a multiple of 13. We use the Principle of Mathematical Induction with $n_0 = 0$. For the basis step, we have $4^{2(0)+1} + 3^{(0)+2} = 4 = 3^2 = 13$, as needed.

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$$
\begin{aligned}
4^{2(k+1)+1} + 3^{(k+1)+2} &= 4^{(2k+1)+2} + 3^{(k+2)+1} \\
&= 4^2(4^{2k+1}) + 4^2\underbrace{(3^{k+2} - 3^{k+2})}_{0} + 3(3^{k+2}) \\
&= 4^2(4^{2k+1} + 3^{k+2}) + 3^{k+2}(-4^2 + 3) \\
&= 16(13t) + 3^{k+2}(-13) \text{ by the induction hypothesis} \\
&= 13(16t - 3^{k+2}).
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4^{2(k+1)+1} + 3^{(k+1)+2} &= 4^{(2k+1)+2} + 3^{(k+2)+1} \\
&= 4^2(4^{2k+1}) + 4^2 \underbrace{(3^{k+2} - 3^{k+2})}_{0} + 3(3^{k+2}) \\
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Example 2.70 (continued)

Example 2.70. For every integer $n \geq 0$, the number $4^{2n+1} + 3^{n+2}$ is a multiple of 13.

Proof (continued). So $4^{2n+1} + 3^{n+2}$ is a multiple of 13 when $n = k + 1$, and the claim holds for $n = k + 1$. By the Principle of Mathematical Induction the proposition holds for all $n \geq 0$, as claimed. □

Theorem 2.71

Theorem 2.71. Every integer $n \geq 2$ is a product of primes numbers.

Proof. We take the proposition $P(n)$ as "integer n is a product of prime numbers." We use the Principle of Mathematical Induction with $n_0 = 2$. For the basis step, we have that $n = 2$, as a prime, is a product of prime numbers, as needed. The induction hypothesis is that t is a product of prime numbers for every t satisfying $2 \leq t \leq k$; here we use the *Strong* Principle of Mathematical Induction.

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