Mathematical Reasoning

Chapter 2. Sets

2.4. Set Inclusion—Proofs of Theorems

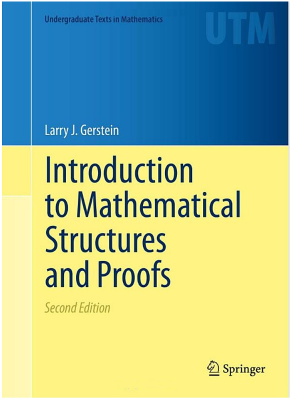

Table of contents

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Theorem 2.15

Theorem 2.15. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Proof. By Definition 2.12, to show $A \subseteq C$ we need to show that $x \in A \Rightarrow x \in C$.

Let $x \in A$. Then since $A \subseteq B$, by Definition 2.12, we have $x \in B$.

Since $B \subset C$, by Definition 2.12, we also have $x \in C$.

Therefore, $A \subseteq C$, as claimed. □

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Theorem 2.17

Theorem 2.17. $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Proof. First, suppose that $A = B$. Then by Definition 2.1, this means $x \in A \Leftrightarrow x \in B$. So $x \in A \Rightarrow x \in B$ and, by Definition 2.12, $A \subseteq B$. Similarly, $x \in B \Rightarrow x \in A$ and, by Definition 2.12, $B \subseteq A$, as claimed.

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Second, suppose $A \subseteq B$ and $B \subseteq A$. Now by Definition 2.12, $A \subseteq B$ means that $x \in A \Rightarrow x \in B$. Similarly, by Definition 2.12 $B \subseteq A$ means that $x \in B \Rightarrow x \in A$. So in this case we have $x \in A \Leftrightarrow x \in B$, and by Definition 2.1 we have $A = B$, as claimed.

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Therefore the two way implication, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$, holds as claimed. □

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Exercise 2.4.8

Exercise 2.4.8. Let A and B be sets. Prove that $A \subseteq B$ if and only if every subset of A is a subset of B.

Proof. First, suppose $A \subseteq B$. Let C be an arbitrary subset of set A: $C \subseteq A$. Then by Theorem 2.15, $C \subseteq B$ (this requires permuting the sets A, B, C here to match with the roles played by sets A, B, C in Theorem 2.15). Since C is an arbitrary subset of set A, then every subseteq of A is a subset of B. That is, $A \subseteq B$ implies that every subset of A is a subset of B, as claimed.

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