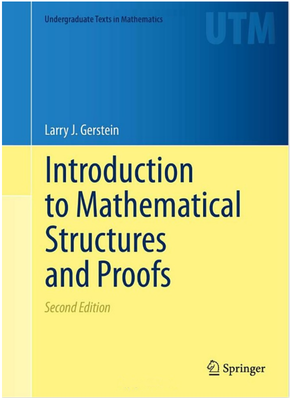
Mathematical Reasoning

Chapter 2. Sets

2.5. Union, Intersection, and Complement—Proofs of Theorems



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Proof. By Definition 2.1, we need to show $x \in A \cap (B \cap C) \Leftrightarrow x \in A \cap (B \cap C)$. We show this with a string of equivalences:

$$\begin{aligned} x \in A \cap (B \cap C) &\Leftrightarrow a \in A \text{ and } x \in (B \cap C) \\ &\Leftrightarrow x \in A \text{ and } (s \in B \text{ and } x \in C) \\ &\Leftrightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Leftrightarrow x \in (A \cap B) \cap C. \end{aligned}$$

So the claim holds. □

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First, let $x \in (A \cap B)'$. Then by Definition 2.21, $x \in U$ and $x \notin A \cap B$. So x is not in both A and B. Hence, x is either (1) not in A or (2) not in B; that is, either (1) $x \in A'$ or (2) $x \in B'$.

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Proof (continued). Second, let $x \in A' \cup B'$. Then by Definition 2.24 either $x \in A'$ or $x \in B'$; that is, either $x \notin A$ or $x \notin B$. Since $A \cap B \subseteq A$ (this follows from the definition of intersection, Definition 2.24), if $x \notin A$ then $x \notin A \cap B$. Similarly, since $A \cap B \subseteq B$, if $x \notin B$ then $x \notin A \cap B$. Now either $x \notin A$ or $x \notin B$ holds, then we have $x \notin A \cap B$; that is, we have $x \in (A \cap B)'$. Since x is an arbitrary element of $A' \cup B'$, then $A' \cup B' \subseteq (A \cap B)'$ by Definition 2.12. Therefore $(A \cap B)' = A' \cup B'$, as claimed. □

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(b) If $Z \subseteq X$ and $Z \subseteq Y$ then $Z \subseteq X \cap Y$.

Proof. (a) Let $w \in X \cup Y$. Then by Definition 2.24, either $w \in X$ or $w \in Y$. If $w \in X$ then $w \in Z$ since $X \subseteq Z$ by hypothesis (by Definition 2.12). If $w \in Y$ then $w \in Z$ since $Y \subseteq Z$ by hypothesis (by Definition 2.12).

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