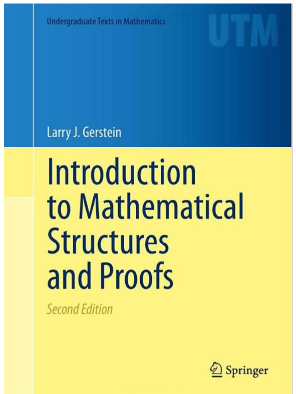
Mathematical Reasoning

Chapter 2. Sets

2.7. The Power Set—Proofs of Theorems



Theorem 2.36

Theorem 2.36. Let A and B be sets. Then:

(a) $\{\varnothing, A\} \subseteq P(A)$

(b) $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$

(c) $P(A) \cup P(B) \subseteq P(A \cup B)$

(d) $P(A) \cap P(B) = P(A \cap B)$

Proof. (a) This follows immediately from Theorem 2.14, which state that $\varnothing \subseteq A$ and $A \subseteq A$. □

(b) First suppose $A \subseteq B$. Let $X \in P(A)$. Then $X \subseteq A \subseteq B$, so that (for the record, by Theorem 2.15) $X \subseteq B$ and hence $X \in P(B)$. Since X is an arbitrary element of $P(A)$ then we have $P(A) \subseteq P(B)$. Second, suppose $P(A) \subseteq P(B)$. Let $x \in A$. Then $\{x\} \subseteq A$ and $\{x\} \in P(A) \subseteq P(B)$, so that $\{x\} \in P(B)$. That is, $\{x\} \subseteq B$ and hence $x \in B$. Since x is an arbitrary element of A, then $A \subseteq B$. Hence $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$, as claimed. □

Theorem 2.36 (continued)

Theorem 2.36. Let A and B be sets. Then:

(c) $P(A) \cup P(B) \subseteq P(A \cup B)$

(d) $P(A) \cap P(B) = P(A \cap B)$

Proof (continued). (c) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (b) we have $P(A) \subseteq P(A \cup B)$ and $P(B) \subseteq P(A \cup B)$. Then by Theorem 2.27(a) $P(A) \cup P(B) \subseteq P(A \cup B)$, as claimed. □

(d) Since $A \cap B \subseteq A$, then by (b) we have $P(A) \subseteq P(A \cup B)$. Since $A \cap B \subseteq B$, then by (b) we have $P(B) \subseteq P(A \cup B)$. So Theorem 2.27(b) implies $P(A \cap B) \subseteq P(A) \cap P(B)$. Conversely, if $X \in P(A) \cap P(B)$ then $X \subseteq A$ and $X \subseteq B$ so that $X \subseteq A \cap B$. That is, $X \in P(A \cap B)$. Since X is an arbitrary element of $P(A) \cap P(B)$ then we have $P(A) \cap P(B) \subseteq P(A \cap B)$. Hence $P(A) \cap P(B) = P(A \cap B)$, as claimed. □

Exercise 2.7.8

Exercise 2.7.8. Let A be a set, and suppose $x \notin A$. Describe $P(A \cup \{x\})$.

Solution. Every subset of A is a subset of $A \cup \{x\}$, so $P(A) \subseteq P(A \cup \{x\})$. Now if $B \in P(A \cup \{x\})$ and B is not a subset of A, then B must include x as an element. For each subset C of A, the set $C \cup \{x\}$ is a subset of $A \cup \{x\}$. So

$$\{B \mid B \in P(A)\} \cup \{C = B \cup \{x\} \mid B \in P(A)\} \subseteq P(A \cup \{x\}).$$

Since the subsets of $A \cup \{x\}$ either exclude x (and so the sets are in $P(A)$) or include x (and so the sets are of the form $C = B \cup \{x\}$ where $B \in P(A)$). Hence

$$P(A \cup \{x\}) = \{B \mid B \in P(A)\} \cup \{C = B \cup \{x\} \mid B \in P(A)\}$$

and this classifies the elements of $P(A \cup \{x\})$. □