Mathematical Reasoning

Chapter 2. Sets

2.7. The Power Set—Proofs of Theorems

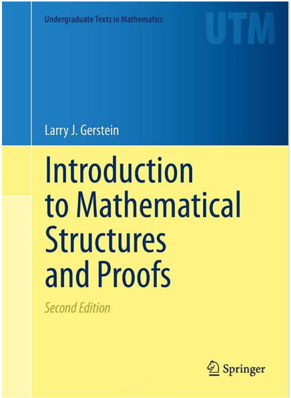

Table of contents

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(a) $\{\varnothing, A\} \subseteq P(A)$

(b) $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$

(c) $P(A) \cup P(B) \subseteq P(A \cup B)$

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Proof (continued). (c) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (b) we have $P(A) \subseteq P(A \cup B)$ and $P(B) \subseteq P(A \cup B)$. Then by Theorem 2.27(a) $P(A) \cup P(B) \subseteq P(A \cup B)$, as claimed. □

(d) Since $A \cap B \subseteq A$, then by (b) we have $P(A) \subseteq P(A \cup B)$. Since $A \cap B \subseteq B$, then by (b) we have $P(B) \subseteq P(A \cup B)$. So Theorem 2.27(b) implies $P(A \cap B) \subseteq P(A) \cap P(B)$.

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(d) Since $A \cap B \subseteq A$, then by (b) we have $P(A) \subseteq P(A \cup B)$. Since $A \cap B \subseteq B$, then by (b) we have $P(B) \subseteq P(A \cup B)$. So Theorem 2.27(b) implies $P(A \cap B) \subseteq P(A) \cap P(B)$. Conversely, if $X \in P(A) \cap P(B)$ then $X \subseteq A$ and $X \subseteq B$ so that $X \subseteq A \cap B$. That is, $X \in P(A \cap B)$. Since X is an arbitrary element of $P(A) \cap P(B)$ then we have $P(A) \cap P(B) \subseteq P(A \cap B)$. Hence $P(A) \cap P(B) = P(A \cap B)$, as claimed. □

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Solution. Every subset of A is a subset of $A \cup \{x\}$, so $P(A) \subseteq P(A \cup \{x\})$. Now if $B \in P(A \cup \{x\})$ and B is not a subset of A, then B must include x as an element. For each subset C of A, the set $C \cup \{x\}$ is a subset of $A \cup \{x\}$. So

$$\{B \mid B \in P(A)\} \cup \{C = B \cup \{x\} \mid B \in P(A)\} \subseteq P(A \cup \{x\}).$$

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Since the subsets of $A \cup \{x\}$ either exclude x (and so the sets are in $P(A)$) or include x (and so the sets are of the form $C = B \cup \{x\}$ where $B \in P(A)$). Hence

$$P(A \cup \{x\}) = \{B \mid B \in P(A)\} \cup \{C = B \cup \{x\} \mid B \in P(A)\}$$

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