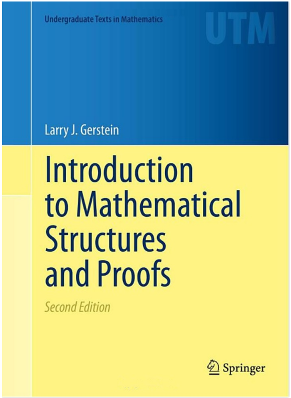
Mathematical Reasoning

Chapter 2. Sets

2.8. Ordered Pairs and Cartesian Products—Proofs of Theorems



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Theorem 2.40. $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

Proof. First, suppose $a = c$ and $b = d$. Then $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$, so that $\{\{c\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. That is, $(a, b) = (c, d)$, as claimed.

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Second, suppose $(a, b) = (c, d)$ (this is Exercise 2.8.8). We consider two subcases. If $a = b$ then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$ and the hypothesis $(a, b) = (c, d)$ means that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, or that $\{\{a\}\} = \{\{c\}, \{c, d\}\}$. From this we must have that $\{c\} = \{a\}$ and $\{c, d\} = \{a\}$, and hence we have $a = c$ and $b = a = d$.

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Theorem 2.45 (a, d, e)

Theorem 2.45. Let A, B, and C be sets. Then:

(a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(d) If A and B are nonempty sets then $A \times B = B \times A \Leftrightarrow A = B$.

(e) If $A_1 \in P(A)$ and $B_1 \in P(B)$, then $A_1 \times B_1 \in P(A \times B)$.

Proof. **(a)** We have the following equivalences:

$$
\begin{aligned}
(x, y) \in (A \cup B) \times C &\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } y \in C \\
&\Leftrightarrow (x \in A \text{ and } y \in C) \text{ or } (x \in B \text{ and } y \in C) \\
&\Leftrightarrow (x, y) \in A \times C \text{ or } (x, y \in B \times C \\
&\Leftrightarrow (x, y) \in (A \times C) \cup (B \times C).
\end{aligned}
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So $(A \cup B) \times C = (A \times C) \cup (B \times C)$, as claimed. $\square$

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So $(A \cup B) \times C = (A \times C) \cup (B \times C)$, as claimed. □

Theorem 2.45 (a, d, e); continued 1

Theorem 2.45. Let A, B, and C be sets. Then:

(d) If A and B are nonempty sets then $A \times B = B \times A \Leftrightarrow A = B$.

Proof (continued). (d) First, suppose $A = B$. Then:

$$\begin{aligned} (x, y) \in A \times B &\Leftrightarrow x \in A \text{ and } x \in B \\ &\Leftrightarrow x \in A = B \text{ and } y \in B = A \\ &\Leftrightarrow (x, y) \in B \times A. \end{aligned}$$

Therefore $A \times B = B \times A$, as claimed.

Conversely, suppose $A \times B = B \times A$. We give an indirect proof (i.e., a proof by contradiction). ASSUME $A \neq B$. Then there is some element in one set that is not in the other; say (without loss of generality) that $A \in A - B$. Let $b \in B$ (which exists since B is nonempty). Then $(a, b) \in A \times B$ and, by hypothesis, $(a, b) \in B \times A$. Hence $a \in B$, CONTRADICTING the assumption that $A \neq B$. Therefore $A \neq B$ is false, and so $A = B$ as claimed. □

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Theorem 2.45. Let A, B, and C be sets. Then:

(d) If A and B are nonempty sets then $A \times B = B \times A \Leftrightarrow A = B$.

Proof (continued). (d) First, suppose $A = B$. Then:

$$\begin{aligned} (x, y) \in A \times B &\Leftrightarrow x \in A \text{ and } x \in B \\ &\Leftrightarrow x \in A = B \text{ and } y \in B = A \\ &\Leftrightarrow (x, y) \in B \times A. \end{aligned}$$

Therefore $A \times B = B \times A$, as claimed.

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Theorem 2.45 (a, d, e); continued 2

Theorem 2.45. Let A, B, and C be sets. Then:

(e) If $A_1 \in P(A)$ and $B_1 \in P(B)$, then $A_1 \times B_1 \in P(A \times B)$.

Proof (continued). (e) First, for $A_1 \in P(A)$ and $B_1 \in P(B)$ (i.e., $A_1 \subseteq A$ and $B_1 \subseteq B$), the claim that $A_1 \times B_1 \in P(A \times B)$ is equivalent to the claim that $A_1 \times B_1 \subseteq A \times B$. So suppose $A_1 \subseteq A$ and $B_1 \subseteq B$. Then:

$$\begin{aligned} (x, y) \in A_1 \times B_1 &\Leftrightarrow x \in A_1 \text{ and } y \in B_1 \\ &\Leftrightarrow x \in A \text{ and } y \in B \text{ since } A_1 \subseteq A \text{ and } B_1 \subseteq B \\ &\Leftrightarrow (x, y) \in B \times A. \end{aligned}$$

Therefore $A_1 \times B_1 \subseteq A \times B$, which is equivalent to the claim. □