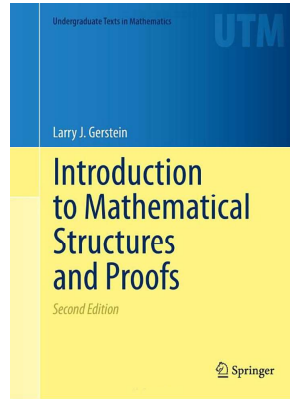


Mathematical Reasoning

Chapter 2. Sets

2.9. Set Decomposition: Partitions and Relations—Proofs of Theorems



Lemma 2.58

Lemma 2.58. If \sim is an equivalence relation and $[x] \neq [y]$ then $[x] \cap [y] = \emptyset$.

Proof. Consider the contrapositive and assume that $[x] \cap [y] \neq \emptyset$. First let $s \in [x]$, so that $s \sim x$. Since $[x] \cap [y] \neq \emptyset$, there is an element $z \in [x] \cap [y]$. That is, $z \sim x$ and $z \sim y$, so that (by reflexivity and transitivity) we have $s \sim x \sim z \sim y$. So (also by transitivity) $s \sim y$, and so $s \in [y]$. Since s is an arbitrary element of $[x]$, then we have $[x] \subseteq [y]$.

Second, let $t \in [y]$, so that $t \sim y$. Since $[x] \cap [y] \neq \emptyset$, there is an element $w \in [x] \cap [y]$. That is, $w \sim x$ and $w \sim y$, so that (by reflexivity and transitivity) we have $t \sim y \sim w \sim x$. So (also by transitivity) $t \sim x$, and so $t \in [x]$. Since t is an arbitrary element of $[y]$, then we have $[y] \subseteq [x]$. Therefore, $[x] = [y]$ and we have shown that the contrapositive of the claim holds, and hence the claim holds. \square

Theorem 2.59

Theorem 2.59. Let \sim be an equivalence relation on a nonempty set S , and let Π be the family of equivalence classes determined by \sim . Then Π is a partition of S . This partition Π is called the partition *induced* by \sim .

Proof. First, represent Π as an indexed set: $\Pi = \{C_i\}_{i \in I}$. By Lemma 2.58, we have $C_i \cap C_j = \emptyset$ for $i \neq j$. So to show that Π is a partition, we need to $\bigcup_{i \in I} C_i = S$. For $x \in \bigcup_{i \in I} C_i$, we have $x \in C_{i'}$ for some $i' \in I$. Since $C_{i'} \subseteq S$, then $x \in S$. Since x is an arbitrary element of $\bigcup_{i \in I} C_i$, then $\bigcup_{i \in I} C_i \subseteq S$. If $s \in S$, then $s \in C_i$ for some $i \in I$; namely $s \in [s]$. So $s \in \bigcup_{i \in I} C_i$ and, since s is an arbitrary element of S , then $S \subseteq \bigcup_{i \in I} C_i$. Therefore, we have $S = \bigcup_{i \in I} C_i$, as needed. \square

Theorem 2.62

Theorem 2.62. Let Π be a partition of the set S . For $x, y \in S$, define $x \sim y$ to mean that x and y belong to the same block of the partition Π . Then \sim is an equivalence relation on S . This is called the equivalence relation *induced* by partition Π .

Proof. Let $\Pi = \{A_i\}_{i \in I}$. Since Π is a partition, we have (by Definition 2.47) that $A_i \cap A_j = \emptyset$ when $i \neq j$, and that $\bigcup_{i \in I} A_i = S$.

If $x \in S$ then $x \in A_i$ for some $i \in I$ and so by the definition of \sim we have $x \sim x$, so that reflexivity holds. For symmetry, suppose $x \sim y$ so that $\{y, x\} = \{x, y\} \subseteq A_i$ for some $i \in I$, and hence $y \sim x$. For transitivity, if $x \sim y$ and $y \sim z$ then $\{x, y\} \subseteq A_i$ and $\{y, z\} \subseteq A_j$ for some $i, j \in I$. But then $y \in A_i \cap A_j$, and therefore we must have $i = j$ since blocks A_i and A_j are disjoint for $i \neq j$. Thus $\{x, y, z\} \subseteq A_i$ and hence $x \sim z$, so that transitivity holds. Since \sim is reflexive, symmetric, and transitive, then it is an equivalence relation, as claimed. \square