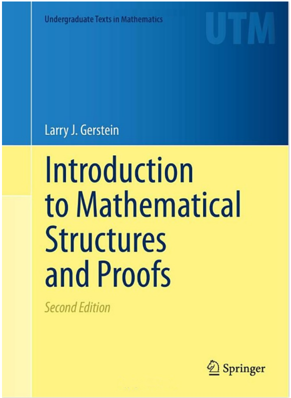
Mathematical Reasoning

Chapter 2. Sets

2.9. Set Decomposition: Partitions and Relations—Proofs of Theorems



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Proof. Consider the contrapositive and assume that $[x] \cap [y] \neq \varnothing$. First let $s \in [x]$, so that $s \sim x$. Since $[x] \cap [y] \neq \varnothing$, there is an element $z \in [x] \cap [y]$. That is, $z \sim x$ and $z \sim y$, so that (by reflectivity and transitivity) we have $s \sim x \sim z \sim y$. So (also by transitivity) $s \sim y$, and so $s \in [y]$. Since s is an arbitrary element of $[x]$, then we have $[x] \subseteq [y]$.

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Second, let $t \in [y]$, so that $t \sim y$. Since $[x] \cap [y] \neq \varnothing$, there is an element $w \in [x] \cap [y]$. That is, $w \sim x$ and $w \sim y$, so that (by reflectivity and transitivity) we have $t \sim y \sim w \sim x$. So (also by transitivity) $t \sim x$, and so $t \in [x]$. Since t is an arbitrary element of $[y]$, then we have $[y] \subseteq [x]$. Therefore, $[x] = [y]$ and we have shown that the contrapositive of the claim holds, and hence the claim holds. □

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Theorem 2.59

Theorem 2.59. Let $\sim$ be an equivalence relation on a nonempty set S, and let Π be the family of equivalence classes determined by $\sim$. Then Π is a partition of S. This partition Π is called the partition *induced* by $\sim$.

Proof. First, represent Π as an indexed set: $\Pi = \{C_i\}_{i \in I}$. By Lemma 2.58, we have $C_i \cap C_j = \varnothing$ for $i \neq j$. So to show that Π is a partition, we need to $\cup_{i \in I} C_i = S$.

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Theorem 2.62. Let Π be a partition of the set S. For $x, y \in S$, define $x \sim y$ to mean that x and y belong to the same block of the partition Π. Then $\sim$ is an equivalence relation on S. This is called the equivalence relation *induced* by partition Π.

Proof. Let $\Pi = \{A_i\}_{i \in I}$. Since Π is a partition, we have (by Definition 2.47) that $A_i \cap A_j = \varnothing$ when $i \neq j$, and that $\cup_{i \in I} A_i = S$.

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If $x \in S$ then $x \in A_i$ for some $i \in I$ and so by the definition of $\sim$ we have $x \sim s$, so that reflexivity holds. For symmetry, suppose $x \sim y$ so that $\{y, x\} = \{x, y\} \subset A_i$ for some $i \in I$, and hence $y \sim x$.

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