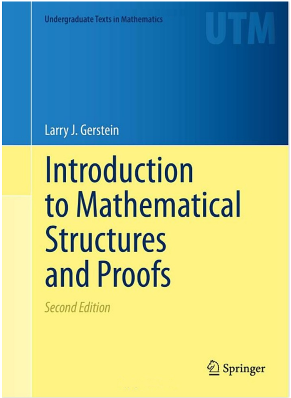
Mathematical Reasoning

Chapter 3. Functions

3.2. Surjections, Injections, Bijections, Sequences—Proofs of Theorems



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$$\begin{aligned} 0 &= x_1^2 + x_1x_2 + x_2^2 = (x_1^2 + x_1x_2) + x_2^2 \\ &= (x_1^2 + x_1x_2 + x_2^2/4) + (x_2^2 - x_2^2/4) = (x_1 + x_2/2)^2 + 3x_2^2/4. \end{aligned}$$

Now both $(x_1 + x_2/2)^2$ and $3x_2^2/4$ are nonnegative and so there sum can only be 0 when both terms are 0. That is, $x_1 = x_2 = 0$ and again $x_1 = x_2$. Therefore, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and f is an injection. □

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Example 3.19

Example 3.19. Let F denote the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. Can the members of f be listed as a sequence? That is, is there a surjection $g : \mathbb{N} \to F$?

Proof/Solution. We give a proof by contraction that no such function g exists. ASSUME function $g : \mathbb{N} \to F$ is surjective. We represent $g(n)$ and g_n, and we then get that the functions in F are $g_1, g_2, \ldots$, of $F = \{g_1, g_2, \ldots\} = \{g_n \mid n \in \mathbb{N}\}$. Define the function $h : \mathbb{N} \to \mathbb{N}$ as $h(n) = g_n(n) + 1$ for all $n \in \mathbb{N}$. Since $h : \mathbb{N} \to \mathbb{N}$ then $h \in F$ and so $h = g_k$ for some $k \in \mathbb{N}$.

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