Mathematical Reasoning

Chapter 3. Functions

3.3. Composition of Functions—Proofs of Theorems

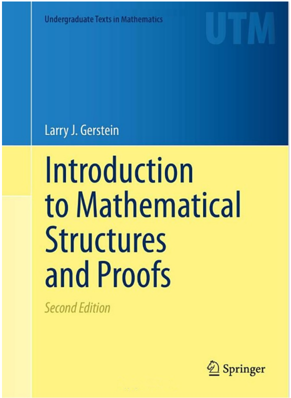

Table of contents

Theorem 3.23. Associative Law of Function Composition

Theorem 3.23. Associative Law of Function Composition.
Given functions $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Notice that $\text{dom}(h \circ (g \circ f)) = \text{dom}((h \circ g) \circ f) = A$. So we need to show that $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ for all $x \in A$. By the definition of function composition, we have for every $x \in A$ that:

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h(g(f(x))) \\ &= h \circ g)(f(x)) = ((h \circ g) \circ f)(x), \end{aligned}$$

and hence $h \circ (g \circ f) = (h \circ g) \circ f$, as claimed. □

Theorem 3.23. Associative Law of Function Composition

Theorem 3.23. Associative Law of Function Composition.
Given functions $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Notice that $\text{dom}(h \circ (g \circ f)) = \text{dom}((h \circ g) \circ f) = A$. So we need to show that $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ for all $x \in A$. By the definition of function composition, we have for every $x \in A$ that:

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h(g(f(x))) \\ &= h \circ g)(f(x)) = ((h \circ g) \circ f)(x), \end{aligned}$$

and hence $h \circ (g \circ f) = (h \circ g) \circ f$, as claimed. □

Theorem 3.24

Theorem 3.24. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions.

(a) If f and g are injections then $g \circ f : A \longrightarrow C$ is an injection.

(b) If f and g are surjections then so is $g \circ f$.

(c) If f and g are bijections then so is $g \circ f$.

Proof. **(a)** Let $a_1, a_2 \in A$ where $(g \circ f)(a_1 = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, which implies $f(a_1) = f(a_2)$ since g is injective. Then $f(a_1) = f(a_2)$ implies $a_1 = a_2$ since f is injective. Therefore, by definition, we have that $g \circ f$ is injective, as claimed. □

Theorem 3.24

Theorem 3.24. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are injections then $g \circ f : A \to C$ is an injection.

(b) If f and g are surjections then so is $g \circ f$.

(c) If f and g are bijections then so is $g \circ f$.

Proof. (a) Let $a_1, a_2 \in A$ where $(g \circ f)(a_1 = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, which implies $f(a_1) = f(a_2)$ since g is injective. Then $f(a_1) = f(a_2)$ implies $a_1 = a_2$ since f is injective. Therefore, by definition, we have that $g \circ f$ is injective, as claimed. □

(b) Let $c \in C$. Since g is surjective, there is some $b \in B$ such that $g(b) = c$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) = g(f(a)) = g(b) = c$, and $g \circ f$ is surjective, as claimed. □

Theorem 3.24

Theorem 3.24. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are injections then $g \circ f : A \to C$ is an injection.

(b) If f and g are surjections then so is $g \circ f$.

(c) If f and g are bijections then so is $g \circ f$.

Proof. **(a)** Let $a_1, a_2 \in A$ where $(g \circ f)(a_1 = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, which implies $f(a_1) = f(a_2)$ since g is injective. Then $f(a_1) = f(a_2)$ implies $a_1 = a_2$ since f is injective. Therefore, by definition, we have that $g \circ f$ is injective, as claimed. □

(b) Let $c \in C$. Since g is surjective, there is some $b \in B$ such that $g(b) = c$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) = g(f(a)) = g(b) = c$, and $g \circ f$ is surjective, as claimed. □

(c) Parts (a) and (b) combine to give (c). □

Theorem 3.24

Theorem 3.24. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are injections then $g \circ f : A \to C$ is an injection.

(b) If f and g are surjections then so is $g \circ f$.

(c) If f and g are bijections then so is $g \circ f$.

Proof. (a) Let $a_1, a_2 \in A$ where $(g \circ f)(a_1 = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, which implies $f(a_1) = f(a_2)$ since g is injective. Then $f(a_1) = f(a_2)$ implies $a_1 = a_2$ since f is injective. Therefore, by definition, we have that $g \circ f$ is injective, as claimed. □

(b) Let $c \in C$. Since g is surjective, there is some $b \in B$ such that $g(b) = c$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) = g(f(a)) = g(b) = c$, and $g \circ f$ is surjective, as claimed. □

(c) Parts (a) and (b) combine to give (c). □

Theorem 3.27

Theorem 3.27. Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof. To show equivalence, first assume that (a) holds. Let $a \in A$. Denote $b = f(a) \in B$. Then by (a), $(g \circ f)(a) = (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a = i_A(a)$. Since a is an arbitrary element of A, we have $g \circ f = i_A$, as claimed.

Theorem 3.27

Theorem 3.27. Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof. To show equivalence, first assume that (a) holds. Let $a \in A$. Denote $b = f(a) \in B$. Then by (a), $(g \circ f)(a) = (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a = i_A(a)$. Since a is an arbitrary element of A, we have $g \circ f = i_A$, as claimed. Let $b \in B$ and denote $a = g(b)$ so that $a = f^{-1}(b)$ and $f(a) = b$. Then by (a), $(f \circ g)(b) = (f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b = i_B(b)$. Since b is an arbitrary element of B, we have $f \circ g = i_B$, as claimed.

Theorem 3.27

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof. To show equivalence, first assume that (a) holds. Let $a \in A$. Denote $b = f(a) \in B$. Then by (a), $(g \circ f)(a) = (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a = i_A(a)$. Since a is an arbitrary element of A, we have $g \circ f = i_A$, as claimed. Let $b \in B$ and denote $a = g(b)$ so that $a = f^{-1}(b)$ and $f(a) = b$. Then by (a), $(f \circ g)(b) = (f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b = i_B(b)$. Since b is an arbitrary element of B, we have $f \circ g = i_B$, as claimed.

Theorem 3.27 (continued 1)

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof (continued). Now suppose that (b) holds. Then for all $a_1, a_2 \in A$ we have that $f(a_1) = f(a_2) \Rightarrow (g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$ since (b) gives $g \circ f = i_A$. Hence f is injective. Also, if $b \in B$ then $g(b) \in A$ and $f(g(b)) = (f \circ g)(b) = b$ since $f \circ g = i_B$. Hence, f is also surjective and so is a bijection, as needed. Therefore $f{-1}$ exists (but we still need to show that $g = f^{-1}$).

Theorem 3.27 (continued 1)

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof (continued). Now suppose that (b) holds. Then for all $a_1, a_2 \in A$ we have that $f(a_1) = f(a_2) \Rightarrow (g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$ since (b) gives $g \circ f = i_A$. Hence f is injective. Also, if $b \in B$ then $g(b) \in A$ and $f(g(b)) = (f \circ g)(b) = b$ since $f \circ g = i_B$. Hence, f is also surjective and so is a bijection, as needed. Therefore $f-1$ exists (but we still need to show that $g = f^{-1}$). The domain of both $g \circ i_B$ and g is set B, and for each $b \in B$ we have $(g \circ i_B)(b) = g(i_B(b)) = g(b)$. So $g \circ i_B = g$. Also the domain of both $i_A \circ f^{-1}$ and f^{-1} is set B, and for each $b \in B$ we have $(i_A \circ f^{-1})(b) = i_A(f^{-1}(b)) = f^{-1}(b)$ (since $f^{-1}(b) \in A$). So $i_A \circ f^{-1} = f^{-1}$.

Theorem 3.27 (continued 1)

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof (continued). Now suppose that (b) holds. Then for all $a_1, a_2 \in A$ we have that $f(a_1) = f(a_2) \Rightarrow (g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$ since (b) gives $g \circ f = i_A$. Hence f is injective. Also, if $b \in B$ then $g(b) \in A$ and $f(g(b)) = (f \circ g)(b) = b$ since $f \circ g = i_B$. Hence, f is also surjective and so is a bijection, as needed. Therefore $f-1$ exists (but we still need to show that $g = f^{-1}$). The domain of both $g \circ i_B$ and g is set B, and for each $b \in B$ we have $(g \circ i_B)(b) = g(i_B(b)) = g(b)$. So $g \circ i_B = g$. Also the domain of both $i_A \circ f^{-1}$ and f^{-1} is set B, and for each $b \in B$ we have $(i_A \circ f^{-1})(b) = i_A(f^{-1}(b)) = f^{-1}(b)$ (since $f^{-1}(b) \in A$). So $i_A \circ f^{-1} = f^{-1}$.

Theorem 3.27 (continued 2)

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Proof (continued). We therefore have

$$\begin{aligned} g &= g \circ i_B \text{ as shown above} \\ &= g \circ (f \circ f^{-1}) \text{ since } f \circ f^{-1} = i_B \\ &= (g \circ f) \circ f^{-1} \text{ by Theorem 3.23, Associative Law} \\ &= i_A \circ f^{-1} \text{ by (b)} \\ &= f^{-1} \text{ as shown above,} \end{aligned}$$

and (a) holds, as claimed. □

Theorem 3.29

Theorem 3.29. Let $f : A \to B$ and $g : B \to C$ be bijections. Then we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. We have by Theorem 3.23, Associative Law of Function Composition:

$$\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ (g \circ f)) \text{ by Theorem 3.23} \\
&= f^{-1}((g^{-1} \circ g) \circ f) \text{ by Theorem 3.23} \\
&= f^{-1} \circ (i_B \circ f) \text{ since } g^{-1} \circ g = i_B \\
&= f^{-1} \circ f \text{ since } i_B \circ f \text{ as shown in the} \\
&\quad \text{proof of Theorem 3.27} \\
&= i_A.
\end{aligned}$$

Similarly, we have $(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_C$. Therefore, by Theorem 3.27 (the "(b) $\Rightarrow$ (a)" part), we have that $g \circ f$ and $f^{-1} \circ g^{-1}$ are inverse functions, as claimed. □

Theorem 3.29

Theorem 3.29. Let $f : A \to B$ and $g : B \to C$ be bijections. Then we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. We have by Theorem 3.23, Associative Law of Function Composition:

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ (g \circ f)) \text{ by Theorem 3.23} \\ &= f^{-1}((g^{-1} \circ g) \circ f) \text{ by Theorem 3.23} \\ &= f^{-1} \circ (i_B \circ f) \text{ since } g^{-1} \circ g = i_B \\ &= f^{-1} \circ f \text{ since } i_B \circ f \text{ as shown in the} \\ &\quad \text{ proof of Theorem 3.27} \\ &= i_A. \end{aligned}$$

Similarly, we have $(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_C$. Therefore, by Theorem 3.27 (the "(b) $\Rightarrow$ (a)" part), we have that $g \circ f$ and $f^{-1} \circ g^{-1}$ are inverse functions, as claimed. □