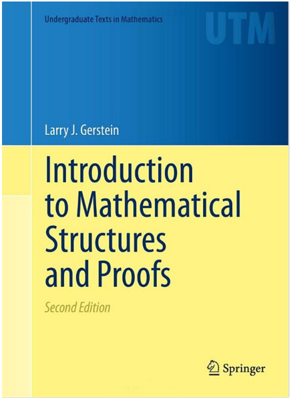
Mathematical Reasoning

Chapter 4. Finite and Infinite Sets

4.1. Cardinality; Fundamental Counting Principles—Proofs of Theorems



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(a) $A \approx A$,

(b) $A \approx B$ implies $B \approx A$, and

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(b) Since $A \approx B$, then there is a bijection $f : A \longrightarrow B$. Since f is a bijection, then $f^{-1} : B \longrightarrow A$ is also a bijection by Note 3.3.A, as needed.

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(c) Since $A \approx B$ and $B \approx C$, then there are bijections $f : A \to B$ and $g : B \to C$. By Theorem 3.24(c), $g \circ f : A \to C$ is a bijection, and hence $A \approx C$, as claimed. □

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Theorem 4.4

Example 4.4. Let X be a set with ten elements, let S be the set of all seven-element subsets of X, and let T be the set of all three-element subsets of X. Then $S \approx T$.

Solution. We establish the claim by giving the bijection, and not by counting the number of subsets in S and T. For $A \in S$, let A' denote the complement of A in X. Since A has seven elements and X has ten elements, then A' has 3 elements; that is, $A' \in T$. Define function $f : S \rightarrow T$ where for each $A \in S$ f maps $A \mapsto A'$. Then f is a bijection and so $S \approx T$, as claimed. □

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Theorem 4.8

Theorem 4.8. Let n and m be nonnegative integers with $n > m$.

(a) There is no injection from $\mathbb{N}_n$ to $\mathbb{N}_m$, and hence $\mathbb{N}_n \not\approx \mathbb{N}_m$.

(b) If A is a set and $\#A = n$, then $\#A \neq m$.

Proof. **(a)** We give an inductive proof on n.

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Proof. (a) We give an inductive proof on n.

For the basis step, let $n = 1$. Then $m = 0$ and there is not an injection from $\mathbb{N}_1$ to $\mathbb{N}_0$ (nor is there even a function from $\mathbb{N}_1$ to $\mathbb{N}_0$; we cannot associate $1 \in \mathbb{N}_1$ with an element of $\mathbb{N}_0 = \varnothing$). Therefore $\mathbb{N}_1 \not\approx \mathbb{N}_0$ and the basis case is established.

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For the induction step, suppose the result is true when $n = k$; that is, if $0 \leq m < k$ there is no injection from $\mathbb{N}_k$ to $\mathbb{N}_m$ (this is the induction hypothesis). ASSUME there is an injection $f : \mathbb{N}_{k+1} \to \mathbb{N}_m$ for some $m < k + 1$. As shown above, there is no function from $\mathbb{N}_{k+1}$ to $\mathbb{N}_0 = \varnothing$ so that we have $m \geq 1$.

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Theorem 4.8. Let n and m be nonnegative integers with $n > m$.

(a) There is no injection from $\mathbb{N}_n$ to $\mathbb{N}_m$, and hence $\mathbb{N}_n \not\approx \mathbb{N}_m$.

Proof (continued). Let g be the bijection that interchanges m with $f(k+1)$ and fixes everything else:

$$g(x) = \begin{cases} f(k+1) & \text{if } x = m \\ m & \text{if } x = f(k+1) \\ x & \text{otherwise.} \end{cases}$$

Then the function $g \circ f : \mathbb{N}_{k+1} \to \mathbb{N}_m$ is an injection by Theorem 3.24(a), and $(g \circ f)(k+1) = g(f(k+1)) = m$. So the restriction $(g \circ f)|_{\mathbb{N}_k}$ is an injection from $\mathbb{N}_k$ to $\mathbb{N}_{m-1}$. But $m - 1 < k$, so the existence of such a function is a CONTRADICTION to the induction hypothesis. So the assumption that there is an injection $f : \mathbb{N}_{k+1} \to \mathbb{N}_m$ for some $m < k+1$ is false, and hence there is no such injection. That is, $\mathbb{N}_{k+1} \not\approx \mathbb{N}_m$. So by The Principle of Mathematical Induction, (a) holds. □

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Theorem 4.8. Let n and m be nonnegative integers with $n > m$.

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Theorem 4.8 (continued 2)

Theorem 4.8. Let n and m be nonnegative integers with $n > m$.

(b) If A is a set and $\#A = n$, then $\#A \neq m$.

Proof (continued). (b) This is easy, given (a). ASSUME $\#A = m$. Then $\mathbb{N}_n \approx A \approx \mathbb{N}_m$, and so $\mathbb{N}_n \approx \mathbb{N}_m$ by Theorem 4.2(c). That is, there is a bijection between $\mathbb{N}_n$ and $\mathbb{N}_m$, a CONTRADICTION to part (a). So the assumption that $\#A = m$ is false, and hence $\#A \neq m$, as claimed. □

Corollary 4.9. The Pigeonhole Principle

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Let A and B be nonempty finite sets, with $\#A > \#B$. Then there is no injection from A to B. Thus for any function $A \to B$, some element in B has at least two preimages.

Proof. Suppose $\#A = n$ and $\#B = m$ where $n > m$. Then by Definition 4.7, there are bijections $f : \mathbb{N}_n \to A$ and $g : B \to \mathbb{N}_m$. ASSUME there is an injection $h : A \to B$. Then the function $g \circ h \circ f : \mathbb{N}_n \to \mathbb{N}_m$ is also an injection by Theorem 3.24(a). But this is a CONTRADICTION to Theorem 4.8(a). So the assumption that there is an injection $h : A \to B$ is false, and so such injection exists, as claimed. □

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Theorem 4.11

Theorem 4.11. Every subset of $\mathbb{N}_n$ is finite, and if $A \subset \mathbb{N}_n$ (that is, A is a proper subset of $\mathbb{N}_n$, $A \subsetneq \mathbb{N}_n$) then $\#A = m$ for some $m < n$.

Proof. We show the second claim that $\#A = m$ for some $m < n$, and the first claim will then follow. We use the Principle of Mathematical Induction on n. For the basis case, with $n = 0$ we have $\mathbb{N}_0 = \varnothing$ and since this has no subset, the result holds vacuously. For the induction hypothesis, suppose the result is true when $n = k$, and consider a subset $A \subset \mathbb{N}_{k+1}$. We now show that $\#A = m$ for some $m \leq k$.

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Case 1. Suppose $k + 1 \notin A$. Then $A \subseteq \mathbb{N}_k$. If $A = \mathbb{N}_k$, then $\#A = k < k + 1$. If $A \subset \mathbb{N}_k$ then the induction hypothesis implies $\#A = m$ for some $m < k < k + 1$. So the result holds for $n = k + 1$ in this case.

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The g is a bijection and therefore $\#A = s+1 \leq k$ (since $s \leq k-1$ then $s+1 \leq k$). So the result holds for $n = k+1$ in this case.

So by the Principle of Mathematical Induction, $\#A = m$ for some $m < n$ as claimed. □

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Theorem 4.12

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(a) Every set containing an infinite set is infinite.

(b) Every set containing an infinite set is infinite.

(c) If $A \subset B$ (that is, $A \subsetneq B$) and B is finite then $\#A < \#B$.

Proof. (a) Suppose $A \subseteq B$ and B is finite. So by Definition 4.7 there is a bijection $f : B \to \mathbb{N}_n$ for some integer nonnegative n. The restricted function $f|_A$ is injective and it is a bijection from A to its range $f(A)$. By Theorem 4.11 we have $f(A) \approx \mathbb{N}_m$ for some integer $m \leq n$. Hence $A \approx f(A) \approx \mathbb{N}_m$, and by Theorem 4.2(c) $A \approx \mathbb{N}_m$ so that A is finite by Definition 4.7, as claimed.

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(b) Let $A \subseteq B$. We have by part (a) that "B finite" $\Rightarrow$ "A finite." The contrapositive of (a) is "A not infinite" $\Rightarrow$ "B infinite," as claimed.

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Theorem 4.13

Theorem 4.13. The set $\mathbb{N}$ of natural numbers is infinite.

Proof. ASSUME $\mathbb{N}$ is finite. Then by Definition 4.7 there is a bijection $f : \mathbb{N} \to \mathbb{N}_m$ for some $m \in \mathbb{N}$. Let n be a natural number such that $n > m$ (this can be done by the *Axiom of Infinity*; see my online notes Introduction to Set Theory on Section 3.1. Introduction to Natural Numbers). Of course $\mathbb{N}_n \subset \mathbb{N}$. Next $f|_{\mathbb{N}_n}$ is an injection from $\mathbb{N}_n$ into $\mathbb{N}_m$. But this CONTRADICTS Theorem 4.8(a) (since $n > m$). So the assumption that $\mathbb{N}$ is finite is false, and hence $\mathbb{N}$ is infinite, as claimed. □

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Theorem 4.14

Theorem 4.14. If A and B are disjoint finite sets, then $A \cup B$ is finite and $\#(A \cup B) = \#A + \#B$.

Proof. Suppose $\#A = m$ and $\#B = n$. Then by Definition 4.7, there exist bijections $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to B$. Define $h : \mathbb{N}_{m+n} \to A \cup B$ by

$$h(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq m \\ g(i - m) & \text{if } m + 1 \leq i \leq m + n. \end{cases}$$

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Proof. Suppose $\#A = m$ and $\#B = n$. Then by Definition 4.7, there exist bijections $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to B$. Define $h : \mathbb{N}_{m+n} \to A \cup B$ by

$$h(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq m \\ g(i-m) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

Since for $a \in A$ we have $f(j_a) = a$ for some $j_a \in \{1, 2, \ldots, m\} = \mathbb{N}_m$ (because $f : \mathbb{N}_m \to A$ is a bijection), and for $b \in B$ we have $g(j_b) = b$ for some $j_b \in \{1, 2, \ldots, \mathbb{N}_n$ (because $g : \mathbb{N}_n \to B$ is a bijection). So for $a \in A$ we have $h(j_a) = f(j_a) = a$, and for $b \in B$ we have $h(j_b + m) = f((j_b + m) - m) = f(j_b) = b$. Now j_a is in $\{1, 2, \ldots, m\}$ and $j_b + m$ is in $\{m+1, m+2, \ldots, m+n\}$, so $h : \mathbb{N}_{m+n} \to A \cup B$ is a surjection.

Theorem 4.14

Theorem 4.14. If A and B are disjoint finite sets, then $A \cup B$ is finite and $\#(A \cup B) = \#A + \#B$.

Proof. Suppose $\#A = m$ and $\#B = n$. Then by Definition 4.7, there exist bijections $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to B$. Define $h : \mathbb{N}_{m+n} \to A \cup B$ by

$$h(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq m \\ g(i - m) & \text{if } m + 1 \leq i \leq m + n. \end{cases}$$

Since for $a \in A$ we have $f(j_a) = a$ for some $j_a \in \{1, 2, \ldots, m\} = \mathbb{N}_m$ (because $f : \mathbb{N}_m \to A$ is a bijection), and for $b \in B$ we have $g(j_b) = b$ for some $j_b \in \{1, 2, \ldots, \mathbb{N}_n$ (because $g : \mathbb{N}_n \to B$ is a bijection). So for $a \in A$ we have $h(j_a) = f(j_a) = a$, and for $b \in B$ we have $h(j_b + m) = f((j_b + m) - m) = f(j_b) = b$. Now j_a is in $\{1, 2, \ldots, m\}$ and $j_b + m$ is in $\{m + 1, m + 2, \ldots, m + n\}$, so $h : \mathbb{N}_{m+n} \to A \cup B$ is a surjection.

Theorem 4.14 (continued)

Theorem 4.14. If A and B are disjoint finite sets, then $A \cup B$ is finite and $\#(A \cup B) = \#A + \#B$.

Proof (continued). Let $c \in A \cup B$. Suppose $h(j) = h(j') = c$. From the definition of h, if $c \in A$ then $h(j) = f(j) = c = f(j')$ and since f is an injection then $j = j'$. Similarly, if $c \in B$ then $h(j) = g(j - m) = c = g(j' - m)$ and since g is an injection then $j - m = j' - m$ or $j = j'$. Notice that we cannot have $c \in A \cap B$ since A and B are disjoint. Therefore, h is an injection.

That is, $h : \mathbb{N}_{m+n} \to A \cup B$ is a bijection and so $\#(A \cup B) = m + n = \#A + \#B$, as claimed. □

Corollary 4.16

Corollary 4.16. If A and B are finite sets (not necessarily disjoint), then $A \cup B$ is finite and

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

Proof. We write $A \cup B$ as a disjoint union of three pairwise disjoint sets: $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$. Then

$$\begin{aligned}
\#(A \cup B) &= \#(A - B) + \#(A \cap B) + \#(B - A) \text{ by Corollary 4.15} \\
&= [\#(A - B) + \#(A \cap B)] + [\#(B - A) + \#(A \cap B)] \\
&\quad -\#(A \cap B) \\
&= \#A + \#B - \#(A \cap B) \text{ by Theorem 4.14,} \\
&\quad \text{since } A = (A - B) \cup (A \cap B) \text{ and } B = (B - A) + (A \cap B),
\end{aligned}$$

as claimed. □

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\#(A \cup B) &= \#(A - B) + \#(A \cap B) + \#(B - A) \text{ by Corollary 4.15} \\
&= [\#(A - B) + \#(A \cap B)] + [\#(B - A) + \#(A \cap B)] \\
&\quad -\#(A \cap B) \\
&= \#A + \#B - \#(A \cap B) \text{ by Theorem 4.14,} \\
&\quad \text{since } A = (A - B) \cup (A \cap B) \text{ and } B = (B - A) + (A \cap B),
\end{aligned}$$

as claimed. □

Theorem 4.17

Theorem 4.17. if $\#A = m$ and $\#B = n$, then $\#(A \times B) = mn$.

Proof. If $A = \varnothing$ then $A \times B = \varnothing$ and the claim follows since $\#\varnothing = 0$. Otherwise, let $A = \{a_1, a_2, \ldots, a_m\}$, say. Then $A \times B = \cup_{i=1}^{m}(\{a_i\} \times B)$ and this is a union of m pairwise disjoint sets, each with the same cardinality as B (namely, $\#(\{a_i\} \times B) = n$). So by Corollary 4.15, $\#(A \times B) = \sum_{i=1}^{m} n = mn$, as claimed. □

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Corollary 4.18

Corollary 4.18. Let $A = \{a_1, a_2, \dots, a_m\}$, and for each i satisfying $1 \leq i \leq m$, let B_i be a set with $\#B_i = n$. Then $\#(\cup_{i=1}^{m}(\{a_i\} \times B_i)) = mn$.

Proof. Let $S_i = \{a_i\} \times B_i$ for $1 \leq i \leq m$. Then the sets S_i are pairwise disjoint (since the first coordinates of pairs in S_i and pairs in S_j are different) and $\#S_i = n$ for each i with $1 \leq i \leq m$. Then by Corollary 4.15, $\#(\cup_{i=1}^{m}(\{a_i\} \times B_i)) = \sum_{i=1}^{m} n = mn$, as claimed. □

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