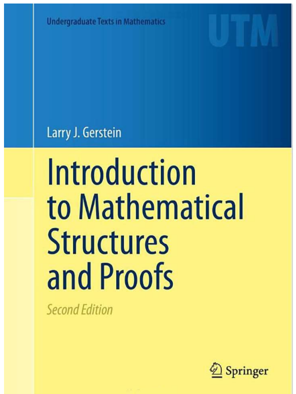
Mathematical Reasoning

Chapter 4. Finite and Infinite Sets

4.2. Comparing Sets, Finite or Infinite—Proofs of Theorems



Lemma 4.25

Lemma 4.25. If A is any set, then $\varnothing$ is an injective function from $\varnothing$ to A.

Proof. This result is true vacuously. First, notice that $\varnothing \subseteq \varnothing \times A$, as needed. Also,

$$\text{if } x \in \varnothing \text{ then } (x, y) \in \varnothing \text{ for exactly one } y \in A$$

is true vacuously (since the hypothesis is false; there are are no $x \in \varnothing$), so that $\varnothing$ really is a function from $\varnothing$ to A (see Definition 3.2). For injectivity, we need to check the implication (see Definition 3.10):

$$\text{if } x_1, x_2 \in \varnothing \text{ and } \varnothing(x_1) = \varnothing(x_2) \text{ then } x_1 = x_2.$$

Again, this is vacuously true since the hypothesis is false (there are no $x_1, x_2 \in \varnothing$). Therefore $\varnothing$ is an injective function from $\varnothing$ to A, as claimed. □

Theorem 4.26

Theorem 4.26. Let A and B be finite sets. Then

(a) $\#A \le \#B \Leftrightarrow$ There is an injection from A to B,

(b) $\#A = \#B \Leftrightarrow A \approx B$, and

(c) $\#A < \#B \Leftrightarrow$ There is an injection but no bijection from A to B.

Proof. Let $\#A = m$ and $\#B = n$. Then there are bijections f and g such that $\mathbb{N}_m \xrightarrow{f} A$ and $\mathbb{N}_n \xrightarrow{g} B$.

(a) If $m \le n$ then there is an injection $j : \mathbb{N}_m \to \mathbb{N}_n$ (namely, the inclusion mapping of Example 3.3(a)). Then the mapping $g \circ j \circ f^{-1} : A \to B$ is an injection (by Theorem 3.24(a)), as claimed.

Conversely, suppose there is an injection $h : A \to B$. Then the mapping $g^{-1} \circ h \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is an injection (by Theorem 3.24(a)), so that $m \le n$ by the contrapositive of Theorem 4.8(a). □

Theorem 4.26 (continued 1)

Theorem 4.26. Let A and B be finite sets. Then

(a) $\#A \le \#B \Leftrightarrow$ There is an injection from A to B,

(b) $\#A = \#B \Leftrightarrow A \approx B$, and

(c) $\#A < \#B \Leftrightarrow$ There is an injection but no bijection from A to B.

Proof (continued). (b) If $m = n$ then there is an bijection $j : \mathbb{N}_m \to \mathbb{N}_n$ (namely, the identity mapping). Then the mapping $g \circ j \circ f^{-1} : A \to B$ is a bijection (by Theorem 3.24(c)) so that $A \approx B$, as claimed.

Conversely, suppose $A \approx B$ so that there is an bijection $h : A \to B$. Then the mapping $g^{-1} \circ h \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is a bijection (by Theorem 3.24(c)), so that $m = n$ by the contrapositive of Theorem 4.8(c) (applied twice). □

Theorem 4.26 (continued 2)

Theorem 4.26. Let A and B be finite sets. Then

(a) $\#A \leq \#B \Leftrightarrow$ There is an injection from A to B,

(b) $\#A = \#B \Leftrightarrow A \approx B$, and

(c) $\#A < \#B \Leftrightarrow$ There is an injection but no bijection from A to B.

Proof (continued). (c) First suppose $m < n$. Then by part (a), there is an injection from A to B. ASSUME there is a bijection $h : A \to B$. Then the map $g^{-1} \circ h \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is a bijection (by Theorem 3.24(c)), but this CONTRADICTS Theorem 4.8(a). So the assumption is false, and there is no bijection from A to B, as claimed.

Second, suppose there is an injection but no bijection from A to B. Then by part (a) we have $\#A \leq \#B$. Since there is no bijection, then $A \not\approx B$ and so by the contrapositive of part (b) we have $\#A \neq \#B$. That is, $\#A < \#B$, as claimed. □

Theorem 4.30

Theorem 4.30. Let A, B, and C be sets. Then

(a) $\#A < \#B < \#C \Rightarrow \#A < \#C$,

(b) $\#A < \#B \leq \#C \Rightarrow \#A < \#C$, and

(c) $\#A \leq \#B < \#C \Rightarrow \#A < \#C$.

Proof. (a) If $\#A < \#B < \#C$ then there is an injection $f : A \to B$ and there is an injection $g : B \to C$. By Theorem 3.24(a) the composition $g \circ f : A \to C$ is an injection and hence $\#A \leq \#C$.

ASSUME $A \approx C$. Then there is a bijection $h : C \to A$. Since $\#B < \#C$ then there is an injection $g : B \to C$. Then the composition $h \circ g : B \to A$ is injective by Theorem 3.24(a), so that $\#B \leq \#A$. But the hypothesis $\#A < \#B$ implies the weaker statement $\#A leq \#B$, so that the Schröder-Bernstein Theorem (Theorem 4.28) then gives that $A \approx B$. But this CONTRADICTS the hypothesis that $\#A < \#B$. So the assumption that $A \approx C$ is false, and we must have $\#A < \#C$, as claimed. □

Theorem 4.31. Cantor's Theorem (I).

Theorem 4.31. Cantor's Theorem (I).
Let S be a set with power set $P(S)$. Then $\#S < \#P(S)$.

Proof. First, if $S = \varnothing$, then $P(S) = \{\varnothing\}$. So $\#S = 0$ and $\#P(S) = 1$ and the claim holds.

Next, suppose $S \neq \varnothing$. Define a function $g : S \to P(S)$ as $g(x) = \{x\}$ for each $x \in S$. Then g is injective since $g(x_1) = g(x_2)$ implies $\{x_1\} = \{x_2\}$ and this implies $x_1 = x_2$. Since g is injective, then $\#S \leq \#P(S)$.

To establish the strict inequality, we show that there is no bijection from S to $P(S)$. ASSUME there is a bijection $f : S \to P(S)$. Then for each $x \in S$ we have $f(x) \in P(S)$. So either $x \in f(x)$ or $x \notin f(x)$. Define set $E \subseteq S$ as $E = \{x \in S \mid x \notin f(x)\}$. Since $E \subseteq S$ then $E \in P(S)$. Since f is onto $P(S)$ then there is some $z \in S$ such that $f(z) = E$. We consider the location of z in relation to set E.

Theorem 4.31. Cantor's Theorem (I); continued.

Theorem 4.31. Cantor's Theorem (I).
Let S be a set with power set $P(S)$. Then $\#S < \#P(S)$.

Proof (continued). $\ldots E = \{x \in S \mid x \notin f(x)\} \ldots$

If $z \in E$ then by the definition of set E, $z \notin f(z) = E$, a CONTRADICTION. If $z \notin E$ then by the definition of set E, $z \in E$, a CONTRADICTION. So the assumption that f is a bijection must be false, and there is no bijection mapping $S \to P(S)$. That is $S \not\approx P(S)$, so that we have $\#S < \#P(S)$, as claimed. □