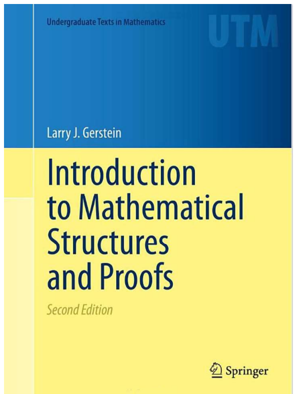
Mathematical Reasoning

Chapter 4. Finite and Infinite Sets

4.3. Countable and Uncountable Sets—Proofs of Theorems



Example 4.34

Example 4.34. The set of integers $\mathbb{Z}$ is countably infinite.

Proof. We know that $\mathbb{N}$ is infinite by Theorem 4.13, and $\mathbb{N} \subset \mathbb{Z}$ so by Theorem 4.12(b) $\mathbb{Z}$ is infinite. Define $f : \mathbb{N} \to \mathbb{Z}$ as

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

First, if n_1 and n_2 are even such that $f(n_1) = f(n_2)$, then $n_1/2 = n_2/2$ and so $n_1 = n_2$. If n_1 is even and n_2 is odd such that $f(n_1) = f(n_2)$, then $n_1/2 = -(n_2-1)/2$ or $n_1 = -n_2 + 1$ or $n_1 + n_2 = 1$; but this cannot happen since $n_1, n_2 \in \mathbb{N}$ and so $n_1 \geq 2$ and $n_2 \geq 1$. If n_1 and n_2 are odd such that $f(n_1) = f(n_2)$, then $-(n_1-1)/2 = -(n_2-1)/2$ and so $-n_1 + 1 = -n_2 + 1$ or $n_1 = n_2$. That is, f is an injection. Now for $m \in \mathbb{Z}$ with $m > 0$, we have $2m \in \mathbb{N}$ and $f(2m) = m$. For $m \in \mathbb{Z}$ with $m \leq 0$, we have $-2m+1 \in \mathbb{N}$ and $f(-2m-1) = -((-2m+1)-1)/2 = m$. That is, f is an injection. So $f : \mathbb{N} \to \mathbb{Z}$ is a bijection, and hence $\mathbb{N} \approx \mathbb{Z}$; that is, $\mathbb{Z}$ is countably infinite, as claimed. □

Theorem 4.35

Theorem 4.35. If A is finite and B is countable then $A \cup B$ is countable.

Proof. First, by Corollary 4.16, if B is also finite then $A \cup B$ is finite and hence countable.

Now suppose that B is countably infinite. We can decompose $A \cup B$ into disjoint sets as $A \cup B = (A - B) \cup B$. Now $A - B \subset A$, so by Theorem 4.12(a) $A - B$ is a finite set. So define finite set $A' = A - B$. If we show that $A' \cup B$ is countable then we have $\cup B$ is countable. Since A' is finite, then there is a bijection $f : A \to \mathbb{N}_n$ for some $n \geq 0$, and since B is countably infinite then there is a bijection $g : B \to \mathbb{N}$. Define function $f : A' \cup B \to \mathbb{N}$ as

$$h(x) = \begin{cases} f(x) & \text{if } x \in A' \\ n + g(x) & \text{if } x \in B. \end{cases}$$

(Notice that f is well defined since A' and B are disjoint.) We claim that h is a bijection.

Theorem 4.35 (continued 1)

Theorem 4.35. If A is finite and B is countable then $A \cup B$ is countable.

Proof (continued). For surjectivity, let $k \in \mathbb{N}$. If $1 \leq k \leq n$ then $k = f(x) = h(x)$ for some $x \in A'$ since $f : A' \to \mathbb{N}_n$ is surjective. If $k \geq n+1$ then $k = n + t$ for some $t \in \mathbb{N}$. Then $k = m + g(x) = h(x)$ for some $x \in B$ since $g : B \to \mathbb{N}$ is surjective.

For injectivity, let $x_1, x_2 \in A' \cup B$. We consider three cases.

Case 1. Suppose $x_1, x_2 \in A'$. Then $h(x_1) = h(x_2)$ implies $f(x_1) = f(x_2)$ and in turn this implies that $x_1 = x_2$ since f is injective.

Theorem 4.35 (continued 2)

Theorem 4.35. If A is finite and B is countable then $A \cup B$ is countable.

Proof (continued).
Case 2. Suppose $x_1, x_2 \in B$. Then $h(x_1) = h(x_2)$ implies $n + g(x_1) = n + g(x_2)$, or $g(x_1) = g(x_2)$, and in turn this implies that $x_1 = x_2$ since g is injective.

Case 3. Suppose $x_i n A'$ and $x_2 \in B$, so that $x_1 \neq x_2$. Then $h(x_1) = f(x_1) \neq n$ and $h(x_2) = n + g(x_2) \geq n + 1$, so that $h(x_1) \neq h(x_2)$. By Note 3.2.B, this means that h is injective in this case.

The three cases combine to show that h is injective. Therefore, $A \cup B = A' \cup B \approx \mathbb{N}$ and $A \cup B$ is countable, as claimed. □

Theorem 4.37

Theorem 4.37. If A and B are countable sets then $A \cup B$ is countable.

Proof. We have $A \cup B = (A - B) \cup B$. By Theorem 4.36(a), $A - B$ is countable since $A - B \subseteq A$. Without loss of generality, we can assume that $A - B$ and B are both countably infinite, since otherwise Theorem 4.16 (when both are finite) and Theorem 4.35 (when exactly one is finite) give the result.

Let set A' be $A' = A - B$ and notice that $A \cup B = A' \cup B$. The integers are countably infinite by Theorem 4.34. The mapping $n \mapsto 2n$ of $\mathbb{Z}$ to the set E of even integers is a bijection and so E is countably infinite. The mapping $n \mapsto 2n - 1$ of $\mathbb{Z}$ to the set E' of odd integers is a bijection and so E' is countably infinite. Since A' and B are countably infinite, then there are bijections $A \xrightarrow{f} E$ and $B \xrightarrow{g} E'$. Now take the union of f and g (as sets of ordered pairs; notice that the domains of f and g are disjoint so that the union actually is a function), $f \cup g$.

Theorem 4.37 (continued)

Theorem 4.37. If A and B are countable sets then $A \cup B$ is countable.

Proof (continued). Since the range of f is E and the range of g is E', then the range of $f \cup g$ is $E \cup E' = \mathbb{Z}$. That is, $f \cup g : A' \cup B \to \mathbb{Z}$ and

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Since f and g are bijections onto E and E', respectively, and $E \cap E' = \varnothing$, then "it follows that" $f \cup g$ is a bijection. So $A \cup B = A' \cup B \approx \mathbb{Z}$ and $A \cup B$ is countably infinite, and the claim holds. □

Theorem 4.39

Theorem 4.39.

(a) $\mathbb{N} \times \mathbb{N}$ is countably infinite.

(b) If A and B are countable then $A \times B$ is countable.

Proof. (a) Define $f : \mathbb{N} \times \mathbb{N}$ by $f(m, n) = 2^m 3^n$. By the Fundamental Theorem of Arithmetic (the unique representation part), f is an injection. Since different values of $m \in \mathbb{N}$ yield different different value of f, then the range of f is infinite so that f is a bijection to an infinite subset of $\mathbb{N}$. A subset of $\mathbb{N}$ is countable by Theorem 4.23(a), so f is a bijection from $\mathbb{N} \times \mathbb{N}$ to a countably infinite set and hence $\mathbb{N} \times \mathbb{N}$ is countably infinite, as claimed. □

Theorem 4.39

Theorem 4.39.

(a) $\mathbb{N} \times \mathbb{N}$ is countably infinite.

(b) If A and B are countable then $A \times B$ is countable.

Proof (continued). (b) he case where both A and B are finite then the claim holds by Theorem 4.17. If A is finite, say $A = \{1_1, a_2, \ldots, a_n\}$, and B is countably infinite then

$$A \times B = (\{a_1\} \times B) \cup (\{a_2\} \times B) \cup \cdots \cup (\{a_n\} \times B).$$

Since $\{a_i\} \times B \approx B$ (as seen by the bijection $(a_i, b) \mapsto b$) then each set $\{a_i\} \times B$ is countably infinite. The claim now holds by Corollary 4.38. Finally, suppose that A and B are both countably infinite. Then there are bijections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Then the mapping from $A \times B$ to $\mathbb{N} \times \mathbb{N}$ given by $(a, b) \mapsto (f(a), g(b))$ is a bijection (as is easily, but maybe tediously, confirmed). So $A \times B \approx \mathbb{N} \times \mathbb{N}$ and so, by part (a), $A \times B$ is countable as claimed. □

Theorem 4.40

Theorem 4.40. The set of rational numbers $\mathbb{Q}$ is countable.

Proof. First write $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$, where $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote the sets of positive and negative rational numbers, respectively. Now $\mathbb{Q}^+ \approx \mathbb{Q}^-$, as seen by the bijection $x \mapsto -x$. We claim that every positive rational number can be written in exactly one way in the form a/b where a and b are positive integers with no common prime factors (this follows from the Fundamental Theorem of Arithmetic; we are representing positive rational numbers as quotients "in lowest terms"). With this notation, we define $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ as $a/b \mapsto (a, b)$. This function is an injection (this is where we need the positive rationals represented in lowest terms), so $\mathbb{Q}^+ \approx f(\mathbb{Q}^+)$ since f is a bijection with its range. By Theorem 4.39(a) $\mathbb{N} \times \mathbb{N}$ is countable, and by Theorem 4.36(a) $f(\mathbb{Q}^+)$ is countable. Hence $\mathbb{Q}^+$ is countable and so is $\mathbb{Q}^-$. Now by Theorem 4.38, $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ is countable, as claimed. □

Theorem 4.41. Cantor's Theorem (II)

Theorem 4.41. Cantor's Theorem (II).
The set of real numbers $\mathbb{R}$ is uncountable.

Proof. We argued in Example 4.5 that every open interval of real numbers is equipotent with $\mathbb{R}$. So we only need to shoe that an open interval is uncountable; we consider $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$. ASSUME $f : \mathbb{N} \to I$ is a bijection. We use the unique decimal representation of the numbers in I mentioned above. With the digits represented by double subscripted a's, we then have

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\ f(2) &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\ f(3) &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\ f(4) &= 0.a_{41}a_{42}a_{43}a_{44}\ldots \\ &\vdots \end{aligned}$$

Theorem 4.41. Cantor's Theorem (II), continued

Theorem 4.41. Cantor's Theorem (II).
The set of real numbers $\mathbb{R}$ is uncountable.

Proof (continued). So a_{ij} is the jth decimal digit of $f(i)$. We again use Cantor's diagonalization method. Define real number $m = 0.m_1m_2m_3m_4\ldots$ by defining the ith decimal digit of m as

$$m_i = \begin{cases} 2 & \text{if } a_{ii} = 1, \\ 1 & \text{if } a_{ii} \neq 1. \end{cases}$$

Then $m \in I$ and for each $i \in \mathbb{N}$ we have $f(i) \neq m$ since $f(i)$ and m differ in the ith decimal place. So $m \notin f(\mathbb{N}) \subset I$ and $f : \mathbb{N} \to I$ is not surjective, a CONTRADICTION. So the assumption that $f : \mathbb{N} \to I$ is a bijection is false and no such bijection exists. That is I, and hence $\mathbb{R}$, is uncountable as claimed. □

Corollary 4.42

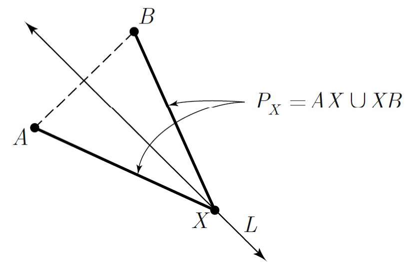
Corollary 4.42. The set of irrational numbers is uncountable.

Proof. Let S be the set of irrational numbers, so that $\mathbb{R} = \mathbb{Q} \cup S$. If S were countable then $\mathbb{R}$ would also be countable by Theorem 4.37. But this contradicts Cantor's Theorem (II) (Theorem 4.41). □

Example 4.43

Example 4.43. This example gives a cute geometric result using an argument based on cardinalities of sets. Since $\mathbb{Q}$ is countable by Theorem 4.10, the set $\mathbb{Q} \times \mathbb{Q}$ is countable (this follows by an argument similar to that for Theorem 4.39 for $\mathbb{N} \times \mathbb{N}$). In the Cartesian plane, $\mathbb{Q} \times \mathbb{Q}$ corresponds to the points having rational coordinates. If A and B are distinct points in the xy-plane and not in $\mathbb{Q} \times \mathbb{Q}$, then A and B can be connected by a path that contains no points in $\mathbb{Q} \times \mathbb{Q}$.

Proof. Let L denote the perpendicular bisector of the line segment AB (see the figure). Then $L \approx \mathbb{R}$. For each point $X \in L$, let P_X denote the path $AX \cup XB$ connecting A to B. We claim that some P_X is a the path that contains no points in $\mathbb{Q} \times \mathbb{Q}$.



Example 4.43 (continued)

Example 4.43. If A and B are distinct points in the xy-plane and not in $\mathbb{Q} \times \mathbb{Q}$, then A and B can be connected by a path that contains no points in $\mathbb{Q} \times \mathbb{Q}$.

Proof (continued). ASSUME that for every $X \in L$ we have $(\mathbb{Q} \times \mathbb{Q}) \cap P_X \neq \varnothing$. Then define $f : L \to \mathbb{Q} \times \mathbb{Q}$ by assigning to each $X \in L$ a point in $(\mathbb{Q} \times \mathbb{Q}) \cap P_X$. Notice from the geometry of the situation, if X and Y are different points on L, then A and B are the only points shared by the paths P_X and P_Y; that is, P_X and P_Y are different. So f is injective by Note 3.2.B, and hence f is a bijection from L to a subset of $\mathbb{Q} \times \mathbb{Q}$ so that L is equipotent with a subset of $\mathbb{Q} \times \mathbb{Q}$. But $L \approx \mathbb{R}$ so L is uncountable (by Cantor's Theorem (II), Theorem 4.41), by Theorem 4.40 and Theorem 4.39(b) $\mathbb{Q} \times \mathbb{Q}$ is countable, and by Theorem 4.36 a subset of a countable set is countable. That is, we have uncountable L is equipotent with a countable set, a CONTRADICTION. So the assumption that every $X \in L$ yields a path from A to B contains a point in $\mathbb{Q} \times \mathbb{Q}$ is false, and some P_X contains no points in $\mathbb{Q} \times \mathbb{Q}$, as claimed. □