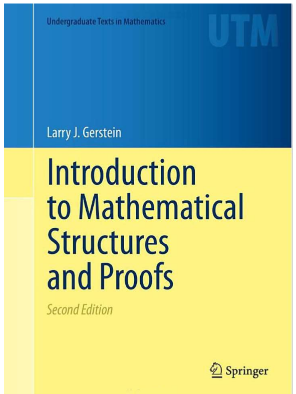
Mathematical Reasoning

Chapter 4. Finite and Infinite Sets

4.4. More on Infinity—Proofs of Theorems



Theorem 4.44

Theorem 4.44. Let S be a set. Then S is infinite if and only if $S \approx S'$ for some $S' \subset S$ (that is, for some $S' \subsetneq S$).

Proof. First, suppose $S \approx S'$ for some $S' \subset S$. ASSUME S is finite, so that $S \approx \mathbb{N}_n$ for some $n \in \mathbb{N}$. Since $S' \subset S$, then by Theorem 4.12(a) S' is finite and so $S' \approx \mathbb{N}_m$ for some $m \in \mathbb{N}$. By Theorem 4.12(c) $m < n$. But we have by Theorem 4.2(c) that $\mathbb{N} \approx \mathbb{N}_m$, which CONTRADICTS Theorem 4.8(a). So the assumption that S is finite is false, and hence S is an infinite set, as claimed.

Conversely, suppose that S is infinite. Then by Theorem 4.36(b), S contains a countably infinite subset $C = \{c_1, c_2, c_2, \ldots\}$. Define $f : S \to S$ as

$$f(x) = \begin{cases} c_{2i} & \text{if } x = c_i \in C \\ x & \text{if } x \notin C. \end{cases}$$

Then f is a bijection of S onto the proper subset $S' = S - \{c_1, c_3, c_5, \ldots\}$. So $S \approx S'$, where S' is a proper subset of S, as claimed. □