Mathematical Reasoning

Chapter 6. Number Theory

6.2. The Integers: Operations and Order—Proofs of Theorems

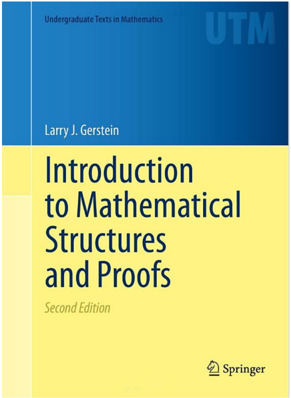

Table of contents

Theorem 6.9

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Proof. (a) First,

$$\begin{aligned} 0 \cdot a &= (0+0) \cdot a \text{ since 0 is the additive identity} \\ &= 0 \cdot a + 0 \cdot a \text{ multiplication distributes over addition by Note 6.2.A.} \end{aligned}$$

Now add $-(0 \cdot a)$ to both sides of this equation (the fact that addition is a binary operation implies that this yields another equation), we get on the 0 on the left-hand side (by the definition of additive inverse) and on the right-hand side we get...

Theorem 6.9

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Proof. (a) First,

$$\begin{aligned} 0 \cdot a &= (0+0) \cdot a \text{ since 0 is the additive identity} \\ &= 0 \cdot a + 0 \cdot a \text{ multiplication distributes over addition by Note 6.2.A.} \end{aligned}$$

Now add $-(0 \cdot a)$ to both sides of this equation (the fact that addition is a binary operation implies that this yields another equation), we get on the 0 on the left-hand side (by the definition of additive inverse) and on the right-hand side we get...

Theorem 6.9 (continued 1)

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Proof (continued). . . .

$$\begin{aligned} -(0 \cdot a) + (0 \cdot a + 0 \cdot a) &= (-(0 \cdot a) + 0 \cdot a) + 0 \cdot a \text{ by associativity of addition} \\ &= 0 + 0 \cdot a \text{ since } -(0 \cdot a) \text{ is the additive inverse of } 0 \cdot a \\ &= 0 \cdot a \text{ since 0 is the additive identity.} \end{aligned}$$

The resulting left-hand and right-hand sides of the first equation give $-c\dot{a} = 0$, as claimed. □

Theorem 6.9 (continued 2)

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Proof (continued). (b) Remember that the "negative sign" represents an additive inverse of an element (which we know to be unique, by Theorem 6.6). Consider:

$$\begin{aligned} a \cdot (-b) + ab &= a(-b + b) \text{ multiplication distributes over addition by Note 6.2.A} \\ &= a \cdot 0 \text{ since } -b \text{ is the additive inverse of } b \\ &= 0 \text{ by part (a).} \end{aligned}$$

So $a \cdot (-b)$ is an additive inverse of ab; that is, $a \cdot (-b) = -ab$, as claimed. □

Theorem 6.9 (continued 3)

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Proof (continued). (c) Assume $ab = ac$ where $a \neq 0$ and add $-ac$ to both sides of this equation to get $ab + (-ac) = ac + (-ac)$, or

$$\begin{aligned} 0 &= ab + (-ac) = ab + a \cdot (-c) \text{ by part (b)} \\ &= a(b + (-c)) \text{ multiplication distributes over addition by Note 6.2.A.} \end{aligned}$$

But $a \neq 0$, so by Note 6.2.A (the fact that $\mathbb{Z}$ has no zero divisors) $a(b + (-c)) = 0$ implies that $b + (-c) = 0$ or (adding c to both sides of this new equation) $b = c$, as claimed. □