Chapter 1. Logic

Note. In this chapter, we discuss mathematics in its broadest sense. We talk about axioms and theorems in Section 1.1. We explore truth tables and their properties in Section 1.2. Conditional statements are the topic of Section 1.3 and proof strategies are given in Section 1.4. The logical equivalence of statements is covered in Section 1.5.

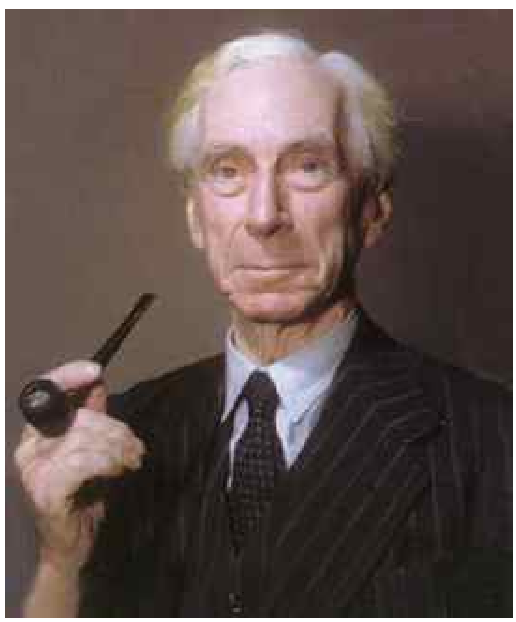
1.1. Statements, Propositions, Theorems

Note. We desire to make statements and give arguments that the statements are true or false. Though we will not formally define "statement," or even "true" or "false," we will operate by the standard that statements are true, false, or neither. That is, we do not have statements that are both true and false.

Note. Our first step is to agree on the kinds of expressions which we take as statements. Second, we classify a collection of statements as true (without proof); these statements are the *axioms* and form our axiomatic system. Finally, we need a system of rules by which we label new statements as "true" or "false"; these rules are our *laws of logic*, *deduction*, *inference*, or *proof*. A statement which has been proved is a *theorem*.

Note. Gerstein (page 2) quotes Bertrand Russell from his autobiography:

> "At the age of eleven, I began Euclid, with my brother as my tutor. . . . From that moment until Whitehead and I finished *Principia Mathematica*, when I was thirty-eight, mathematics was my chief interest, and my chief source of happiness. . . . I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first I refused to accept them unless my brother could offer me some reason for doing so, but he said: 'If you don't accept them we cannot go on,' and as I wished to go on, I reluctantly admitted them pro tem. The doubt as to the premisses of mathematics which I felt at that moment remained with me, and determined the course of my subsequent work."



Bertrand Russell (May 18, 1872–February 2, 1970)

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Note. Russell's *Principia Mathematica*, which he coauthored with Alfred North Whitehead, is a three-volume work on the foundations of mathematics. A goal of the work is to minimize the number of axioms and to express the proofs of theorems in terms of symbolic logic. This project was inspired in part by the paradoxes that were discovered in logic and set theory around the year 1900. For a description of Russell's paradox, see my online notes for Set Theory on Section 1.1. Introduction to Sets and for Analysis 1 (MATH 4217/5217) on Section 1.1. Sets and Functions. When Russell expresses disappointment over starting with the foundational axioms, he is looking for an *a priori* starting point (Immanuel Kant describes *a priori* knowledge in his 1787 *Critique of Pure Reason* as "knowledge that is absolutely independent of all experience"). His expressed doubt about the axioms implies his motivation to minimize such assumptions.

Note. Probably the first axiomatic system we all become familiar with is that of Euclidean geometry. We think of (plane) Euclidean geometry in terms of the points, lines, triangles, circles, etc. that we draw on sheets of paper. But purely as a mathematical construct, many of these terms are left undefined! This is the case for the term *set* in set theory; we can only give definitions of terms using other terms, so at some point we must deal with undefined terms. The axioms of geometry (that is, the statements about the objects of geometry which we take to be true without proof) are what gives meaning to the objects. Then following the laws of logic, we derive other statements (i.e., theorems) from the axioms. We may turn to drawings on a sheet of paper to help us understand the axioms or the theorems, but strictly speaking the drawings are *not* lines or points. Lines and points are *ideas* and have no existence outside of the mind! However, we have accepted our terms and axioms based on our expectation of creating a mathematical model that mimics our experience with the "real world" of lines and points on a flat sheet of paper. So, we could argue, that the axioms are motivated by our experience (an argument that might sadden Kant)! However, we can carefully change the axioms in a way that contradicts our experience and still, potentially, produce an alternative axiomatic system (such as that given by the non-Euclidean geometry of hyperbolic geometry). For more details on axiomatic systems, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.3. Axiomatic Systems; for information on the consistency, independence, and completeness of axiomatic systems see Section 1.6. Completeness and Categoricalness (this includes a brief discussion of the work of Kurt Gödel [April 28, 1906–January 14, 1978] on these topics).

Example 1.1. Consider a mathematical system in which a statement is a string (sequence) of the symbols a, b, and S. For example, some statements are $abaaS$, $bSSaba$, $baaaab$, and SSS. We start with one axiom: the statement S. The laws of logic which we follow are the two rules: (1) a statement obtained from a true statement be replacing an S with aSb is also true; (2) a statement obtained from a true statement be deleting an S and closing up any resulting space is also true. We get a "two column proof" (which you probably encountered in high school geometry) that $aaabbb$ is a theorem:

S	Axiom
aSb	Rule 1
$aaSbb$	Rule 1
$aaaSbbb$	Rule 1
$aaabbb$	Rule 2

More generally, we could prove that $\underbrace{aa\cdots abb\cdots b}_{n}$ is a theorem (by applying Rule 2, we can deduce this theorem for $n = 1$ and $n = 2$ using parts of the previous proof).

Note. In a mathematical system, statement that has been shown to be true or shown to be false is a *proposition* of the system and the label 'true' or 'false' is the *truth value* of the proposition. Gerstein does not address additional possible statuses of statements. One such status is that of meaninglessness. An example of a meaningless statement is: "Love is blue." This statement is neither true nor false (it does not have a truth value). Kurt Gödel addressed this by dealing with "well-formed formulas." A meaningful statement that does not have a truth value can could turn out to be true or false by finding a proof of the statement or its negation. But it could also be *undecidable*; that is, it cannot be assigned a truth value based on the collection of accepted axioms. It then would be possible to add the statement as a new axiom (or to add its negation as a new axiom). An example of an undecidable statement is the Continuum Hypothesis. This is related to the existence of a set of real numbers that is larger then the set of natural numbers and smaller than the set of real numbers (more precisely, this is related to the cardinalities of sets). It is not only unknown if such a set exists, it is *unknowable* (within the standard axioms of the real numbers)! The Continuum Hypothesis states that no such set exists. For more on this, see the above-mentioned notes on Section 1.6. Completeness and Categoricalness and my more general online notes for Great Ideas in Science (BIOL 3018) on Introduction to Math Philosophy and Meaning.

Example 1.2(b). Consider the mathematical system of elementary arithmetic. A *prime number* is an integer greater than 1 that cannot be expressed as a product of two smaller positive integers. In practice, it can be difficult to determine whether a given integer is prime. For example, numbers of the form $F_n = 2^{2^n} + 1$ are *Fermat numbers*. The 33rd Fermat number is $F_{33} = 2^{2^{33}} = 2^{8589934592} + 1$. It has 2,585,827,973 decimal digits. The statement "F_{33} is prime" is a proposition since it is either true or false, but neither this statement nor its negation is a theorem because its truth value is currently unknown.

Note. The use of the terms "theorem" and "proposition" as given above are not universal. It is common use the word *theorem* for a true statement that is of major importance, and to call a theorem of relatively less importance a *proposition* (though I have limited experience with this use of the term). A theorem of interest primarily for its use in proving a more important theorem is often called a *lemma*. A theorem that follows easily from a previous theorem is a *corollary*.

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