1.2. Logical Connectives and Truth Tables

Note. In this section we introduce the idea of negation of a statement, the logical connectives of disjunction, conjunction, and exclusive conjunction ("`xor`"). We translate verbal statements into symbolic statements and develop truth tables for the symbolic statements.

Definition. The *negation* of a statement is formed be using the logical term "not" in modifying the statement. If P is a statement, then the negation of P is denoted $\sim P$ (read as "not P").

Note. Gerstein gives a few examples of the negation of statements. If P is the statement "Rosco is smiling" then $\sim P$ is the statement "Rosco is not smiling." If P involves numerical information, then the negation can be trickier. If P is the statement "No man is an island" then $\sim P$ is (for example) "At least one man is an island." Equalities are easy to negate. If P is $2 - 7 = 4$ then $\sim P$ is $2 - 7 \neq 4$.

Definition. A *truth table* is a list statements and negations or combinations of statements (to be defined below) which presents all possible truth values of the statements and the resulting truth values of the negations or combinations of statements.

Note. As an example of a truth table, consider the statement P and its negation $\sim P$:

P	$\sim P$
F	T
T	F

Definition. Two statements P and Q can be joined together with the logical connective "and" to make a new statement "P and Q," denoted $P \wedge Q$. This is the *conjunction* of P and Q. $P \wedge Q$ is true when both P and Q are true, and is false otherwise.

Note. The truth table for $P \wedge Q$ requires four rows (for the four possible cases of truth values of P and Q) and is:

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Note. Expressions that become statements when the letters of the expressions (called *statement letters* or *sentential variables*) are replaced with explicit statements are *statement forms* or *sentential forms*.

Example 1.6. We can use negation and conjunction, along with parentheses, the create several statement forms based on statement letters P, Q, and R:

$$(P \wedge (\sim Q)) \wedge R, \quad \sim (\sim Q), \quad \sim P((\sim Q(\wedge R), \quad Q \wedge (\sim (\sim (\sim (\sim P)))).$$

Of course the syntax of statements requires that meaningful statements are created. For example, $\wedge \wedge P \wedge$ and $P \sim Q$ are not statements. Notice that a truth table for statements based on statement variables P, Q, and R requires $2^3 = 8$ rows in order to cover all possible cases of truth values for the 3 variables. In general, a truth table for statements based on n statement variables requires 2^n rows. Gerstein presents the rows by starting with all variables having a truth value of F, concluding with all variables having a truth value of T, and walking through the cases in an order corresponding to binary counting from 0 to $2^n - 1$ with 0 represented by F and 1 represented by T. □

Definition. Two statements P and Q can be joined together with the logical connective "or" to make a new statement "P or Q," denoted $P \vee Q$. This is the *disjunction* of P and Q. $P \vee Q$ is true when either P or Q are true, and is false otherwise.

Note. The truth table for $P \vee Q$ is:

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Note 1.2.A. Notice that disjunction is an "inclusive" version of the idea of "or." The "exclusive" version (sometimes referred to as `xor` in various programming languages; it will be denoted $\veebar$ in Section 1.6. Application: A Brief Introduction to Switching Circuits) is true when *exactly one* of P and Q is true, and is false otherwise so that the truth table is:

P	Q	P xor Q
F	F	F
F	T	T
T	F	T
T	T	F

Example 1.10. Consider the statement: "I will go to the movies on Monday or Tuesday, but not on both says." Represent it by a sentential form.

Solution. Introduce the statement letters (or "sentential variables")

P: I will go to the movies on Monday.

Q: I will go to the movies on Tuesday.

Then "I will go to the movies on Monday or Tuesday" becomes the sentential form $P \vee Q$. Also, "but not on both days" (or "I will not go to the movies on both Monday and Tuesday") becomes $\sim (P \wedge Q)$. We translate the term "but" into something like "and" (conjunction) so that the given statement becomes $(P \vee Q) \wedge \sim (P \wedge Q)$. In fact, this could be given by the exclusive or simply as: P xor Q. $\square$

Example 1.11. Let K be the following sentential form:

$$\sim (P \wedge Q) \wedge \underbrace{(P \wedge (\sim Q \vee (\sim P \vee Q)))}_{J}.$$

Under what truth values for Propositions P and Q is the proposition represented by K true?

Solution. We construct a truth table, but introduce several new statements so that in each column we are simply negative, conjuncting, or disjuncting previously known statements:

P	Q	$P \wedge Q$	$\sim Q$	$\sim P$	$\sim P \vee Q$	$\sim Q \vee (\sim P \vee Q)$	J	$\sim (P \wedge Q)$	K
F	F	F	T	T	T	T	F	T	F
F	T	F	F	T	T	T	F	T	F
T	F	F	T	F	F	T	T	T	T
T	T	T	F	F	T	T	T	F	F

We have that K is true when P is true and Q is false, and is negative otherwise. In fact, this is the same truth values of the statement $P \wedge \sim Q$.

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