1.3. Conditional Statements

Note. In this section we consider statements of the form "If P then Q." We address sentential forms involving this statement using truth tables.

Definition. For P and Q propositions, a statement of the form "If P then Q" is a *conditional statement*. We denote this as $P \Rightarrow Q$. Proposition P is the *antecedent* or *hypothesis*, and Q is the *consequent* or *conclusion*.

Example. For nostalgic reasons, let's consider an example from my 10th grade geometry class that I took in 1978–79. Let P be the statement "I find a five-dollar bill" and Q be the statement "I'll take you to the movies" (see my online notes for High School Geometry on Section 2-6. Conditionals). The prices reveal the age of this example! There are four possible situations with respect to P and Q:

1. I find a five-dollar bill and I take you to the movies.

2. I find a five-dollar fill and do not take you to the movies.

3. I do not find a five-dollar bill and I take you to the movies.

4. I do not find a five dollar bill and I do not take you to the movies.

Notice that these correspond respectively to (1) both P and Q are true, (2) P is true and Q is not, (3) P is false and Q is true, and (4) P is false and Q is false. We want to associate a truth value with the conditional statement "If P then Q." Certainly the truth value of $P \Rightarrow Q$ is T in (1) and F in (2). Notice that if I do not find a five-dollar bill (that is, if P is false) then the statement "If I find a five-dollar bill, then I'll take you to the movies" seems largely irrelevant! We certainly cannot say that it is false. So we take the truth value as T in case (3). Similarly, the truth value is T in case (4).

Note. Gerstein approaches this more formally. He says that the conditional statement $P \Rightarrow Q$ as: "Truth of P and falsity of A do not coexist." This can be represented symbolically as $\sim (P \wedge \sim Q)$. In addition to $P \Rightarrow Q$, the statement is also sometimes denoted $P \rightarrow Q$, $P \supset Q$, and $Q \Leftarrow P$. The truth table is

P	Q	$P \Rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Sometimes we read $P \Rightarrow Q$ as "P is sufficient for Q," "Q is necessary for P," P only if Q," and "Q if P."

Example 1.17. Construct a truth table for the following propositional form J:

$$P \Rightarrow \underbrace{((\sim Q \Rightarrow P) \wedge (Q \vee \sim P))}_{K}.$$

Solution. In the truth table, we include all "fundamental propositions."

P	Q	$\sim P$	$\sim Q$	$\sim Q \Rightarrow P$	$Q \vee \sim P$	K	J
F	F	T	T	F	T	F	T
F	T	T	F	T	T	T	T
T	F	F	T	T	F	F	F
T	T	F	F	T	T	T	T

Notice that J has the same truth values as $P \Rightarrow Q$.

Definition. The implication $Q \Rightarrow P$ is the *converse* of the implication $P \Rightarrow Q$. The proposition $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is the *biconditional proposition* (or "material equivalence") and is denoted $P \Leftrightarrow Q$. If $P \Leftrightarrow Q$ is true, then P and Q are *equivalent propositions*.

Note. The truth table for the biconditional proposition is:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

So $P \Leftrightarrow Q$ is true only when P and Q have the same truth values (thus the term "equivalent propositions" in this case).

Exercise 2.3.18. A *tautology* is a sentential form that becomes a true proposition whenever the letters in the expression are replaced by actual propositions. For example, the expressions $P \vee \sim P$ and $(P \vee Q) \Leftrightarrow (Q \vee P)$ are both tautologies. Use truth tables to determine which of the following are tautologies.

(a) $P \Rightarrow ((\sim P) \Rightarrow Q)$

(c) $((P \Rightarrow Q) \Leftrightarrow Q) \Rightarrow P$

(a) Solution. The truth table is:

P	Q	$\sim P$	$\sim P \Rightarrow Q$	$P \Rightarrow ((\sim P) \Rightarrow Q)$
F	F	T	F	T
F	T	T	T	T
T	F	F	T	T
T	T	F	T	T

Since $P \Rightarrow ((\sim P) \Rightarrow Q)$ always has a truth value of T, then it is a tautology. □

(b) Solution. The truth table is:

P	Q	$P \Rightarrow Q$	$((P \Rightarrow Q) \Leftrightarrow Q)$	$((P \Rightarrow Q) \Leftrightarrow Q) \Rightarrow P$
F	F	T	F	T
F	T	T	T	F
T	F	F	T	T
T	T	T	T	T

When P is F and Q is T, the truth value of $((P \Rightarrow Q) \Leftrightarrow Q) \Rightarrow P$ is F, so that it is not a tautology. □

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