1.5. Logical Equivalence

Note. In this section we consider sentential forms with the same truth values. We consider conditions under which sentential forms necessarily have the same truth values and introduce some new notation.

Note. We introduced the idea of a tautology in Exercise 2.3.18 (see the online notes for this class on Section 1.3. Conditional Statements). We now offer another formal definition of a tautology.

Definition. A statement form is a *tautology* if every substitution of propositions for its sentential variables yields a true proposition.

Example 1.25(d). We can use truth tables to determine if a statement is a tautology. Consider the statement $((P \Rightarrow Q) \wedge P) \Rightarrow Q$. The truth table is:

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Since statement $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ is true regardless of the truth value of P and Q (i.e., $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ is true for every substitution of propositions P and Q), then $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ is a tautology.

Definition 1.26. Suppose S_1 and S_2 are sentential forms. Then S_1 and S_2 are *logically equivalent* if the sentential form $S_1 \Leftrightarrow S_2$ is a tautology. The form $S_1 \Leftrightarrow S_2$ is then called a *logical equivalence.*

Example 1.27. Let S_1 be the sentential form $\sim (P \wedge Q)$ and let S_2 be the sentential form $\sim P \vee \sim Q$. We claim that S_1 and S_2 are logically equivalent. To see this, we consider the truth table for $S_1 \Leftrightarrow S_2$:

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	S_1	S_2	$S_1 \Leftrightarrow S_2$
F	F	T	T	F	T	T	T
F	T	T	F	F	T	T	T
T	F	F	T	F	T	T	T
T	T	F	F	T	F	F	T

Since $S_1 \Leftrightarrow S_2$ has truth value T regardless of the truth value of P and Q then S_1 and S_2 are logically equivalent (and $S_1 \Leftrightarrow S_2$ is a logical equivalence).

Note. Notice that the sentential forms $\sim (P \vee Q)$ and $\sim (P \wedge A)$ are also logically equivalent. Consider the truth table:

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	F	F
T	T	T	F	F	T	F

Since the truth values of $\sim (P \vee Q)$ and $\sim (P \wedge Q)$ are the same, regardless of the truth values of P and Q, then $\sim (P \vee Q)$ and $\sim (P \wedge Q)$ are logically equivalent.

Definition. The logical equivalence of $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ of Example 1.27 and the logical equivalence of $\sim (P \vee Q)$ and $\sim (P \wedge Q)$ of the previous example are called *De Morgan's Laws*.

Note. We denote the logical equivalence of sentential forms S_1 and S_2 as $S_1 \equiv S_2$. Then De Morgan's Laws can be stated (with more parentheses) as

$$\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q) \text{ and } \sim (P \wedge Q) \equiv (\sim P) \wedge (\sim Q).$$

Note. Now if $S_1 \equiv S_2$, then truth tables (and the truth values of sentential forms) are unaffected by replacing S_1 with S_2. This leads to the following "principle."

Replacement Principle. If $S_1 \equiv S_2$ and if in some sentential form S and occurrence of S_1 is replaced by S_2, the resulting sentential form is logically equivalent of S.

Note/Definition. The following are logical equivalences, which are straightforward to establish with truth tables:

$$\left.\begin{aligned} P \wedge (Q \wedge R) &\equiv (P \wedge Q) \wedge R \\ P \vee (Q \vee R) &\equiv (P \vee Q) \vee R \end{aligned}\right\} \textit{Associative Laws}$$

$$\left.\begin{array}{r} P \Leftrightarrow Q \equiv Q \Leftrightarrow P \\ P \wedge Q \equiv Q \wedge P \\ P \vee Q \equiv Q \vee P \end{array}\right\} \textit{Commutative Laws}$$

$$\left.\begin{array}{r} P \wedge P \equiv P \\ P \vee P \equiv P \end{array}\right\} \textit{Idempotency Laws}$$

$$\left.\begin{array}{r} P \wedge (P \vee Q) \equiv P \\ P \vee (P \wedge Q) \equiv P \end{array}\right\} \textit{Absorption Laws}$$

$$\left.\begin{array}{r} P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \end{array}\right\} \textit{Distributive Laws}$$

$$\sim (\sim P) \equiv P \quad \textit{Law of Double Negation.}$$

Note. Informally, one of our objectives is to replace a sentential form with a simpler (and shorter) equivalent sentential form. This can lead to a clearer (and more efficient) understanding.

Exercise 1.5.4. We introduce a single logical connective that can serve by itself as an adequate set of connectives. We write $P \uparrow Q$ as an abbreviation for $\sim (P \wedge Q)$. This new connective $\uparrow$ is called the *Sheffer stroke*. Use truth tables to verify the logical equivalences **(a)** $\sim P \equiv P \uparrow P$ and **(c)** $P \wedge Q \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$.

(a) Consider the truth table:

P	$\sim P$	$P \wedge P$	$P \uparrow P = \sim (P \wedge P)$
F	T	F	T
T	F	T	F

So $\sim P$ and $P \uparrow P$ have the same truth values and hence are equivalent, as claimed. $\square$

(c) Consider the truth table:

P	Q	$P \wedge Q$	$P \uparrow Q = \sim (P \wedge Q)$	$(P \uparrow Q) \wedge (P \uparrow Q)$	$\sim ((P \uparrow Q) \wedge (P \uparrow Q))$
F	F	F	T	T	F
F	T	F	T	T	F
T	F	F	T	T	F
T	T	T	F	F	T

So $P \wedge Q$ and $(P \uparrow Q) \uparrow (P \uparrow Q) = \sim ((P \uparrow Q) \wedge (P \uparrow Q))$ have the same truth values and hence are equivalent, as claimed. $\square$

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