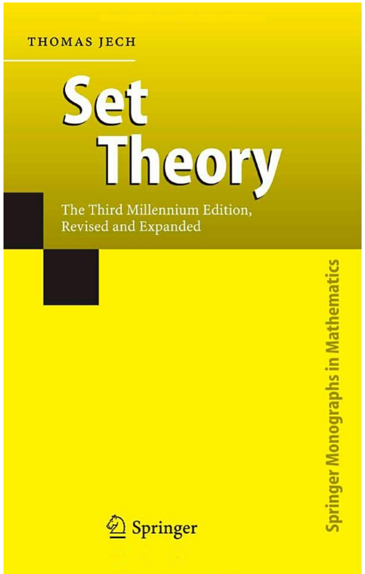
Chapter 2. Sets

Note. This chapter gives a brief survey of what might be called "naive set theory." I have online notes for a (also brief) course on Naive Set Theory based on Paul Halmos' book of the same name (Princeton: D. Van Nostrand Company [1960], reprinted by Dover Publications [2017]). I have online notes on a more formal/axiomatic approach to set theory at the senior/introductory graduate level for Introduction to Set Theory based on Karel Hrbacek and Thomas Jech's *Introduction to Set Theory* (Marcel Dekker [1984]). Thomas Jech is a big name in set theory. His 700-plus page *Set Theory, The Third Millennium Edition, Revised and Expanded*, Springer-Verlag (1997, revised and corrected 2006), is an industry standard for a graduate-level set theory class.



Unfortunately, ETSU does not offer a formal class in set theory. The topics of set theory are covered in this class, Mathematical Reasoning (MATH 3000), in some detail, and in Analysis 1 (MATH 4217/5217) in passing (see Section 1.1. Sets and Functions).

2.1. Fundamentals

Note. In this section we introduce sets an the notation we use to discuss sets. We define equal sets, define sets using a property P and the Axiom of Separation, and define the empty set.

Note. We have an intuitive idea of what a *set* is. Informally, it is a "collection" of "objects" (the objects are called *elements* of the set). But if we use the term "collection" to define a set, then we need to define "collection." Since we can only define new terms by means of previously defined terms, then we must have some starting-point of undefined (or, as Gerstein calls them *primitive*) terms. We take the words *set* and *element* as undefined terms. For verbal variety, we may also use the terms "family" and "collection" in place of set, and use the term "member" is use of element. We relate the elements of the set by using the term "belongs"; that is, an element of a set *belongs* to the set. See my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.3. Axiomatic Systems for a discussion of undefined terms in the setting of geometry.

Note. The two basic problems of this section are (1) to determine, based on the on/off status of the power sources on the left, whether or not power flows out of the single wire on the right side, and (2) to find a configuration of a minimal number of gates that will result in the same output to the right (based on the input from the left) as a given network of gates. We deal with these problems by relating them to sentential forms and sentential variables. The wires on left become the variables and on/off is associated with T/F. A truth table then allows us to solve problem (1) and if we can find a logically equivalent form of the sentential form with the fewest logical connectives then we solve problem (2).

Note/Definition. We denote the fact that x is an element of set A as $x \in A$. The symbol $\in$ is the *membership symbol*. If x is not an element of set A, then we write $x \notin A$. We usually use upper case letters to represent sets and lower case letters to represent elements of sets (as is largely standard in mathematics text books).

Definition 2.1. Sets A and B are *equal*, denoted $A = B$, if they have the same members. That is, $A = B$ means that $x \in A \Leftrightarrow x \in B$. If sets A and B are not equal, we write $A \neq B$.

Note. We may present a small set by listing its elements. For example, the set with 1 and $\sqrt{2}$ as its elements is the set $\{1, \sqrt{2}\}$. Notice that a candidate element of a set is either in the set or not. It is not partially in the set, it is not in the set multiple times, and it does not have a "position" in a set (so even if we list the elements of a set, the location in the list of each element is irrelevant). There are *multi-sets* which can have an element repeated in the set, but that is not of interest here.

Note. We will often use a property P to completely characterize the elements of a set A. We write $A = \{x \mid P(x)\}$, read "A is equal to the set of all x such that property P holds for x." Unless we have a clearly defined universal set from which all elements are chosen, then property P must indicate what the set elements are (such as real numbers) and indicate a property of the set elements (such as being non-negative). For example, we might have property $P(x)$ as "x is a positive integer less than or equal to 5." In this case, $A = \{x \mid P(x)\} = \{x \mid x \text{ is a positive integer and } x \leq 5\} = \{1, 2, 3, 4, 5\}$.

Definition. Some common sets of numbers with which you are familiar are:

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$, the *natural numbers* or *positive integers*

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the *integers*

$\mathbb{Q} = \{x \mid x = a/b \text{ for some } a, b \in \mathbb{Z}, b \neq 0\}$, the *rational numbers*

$\mathbb{R}$, the *real numbers.*

Note. Some books include 0 as a natural number. Gerstein also calls the integers the "whole numbers," but the whole numbers are more traditionally the set of non-negative integers $\{0, 1, 2, 3, 4, \ldots\}$. It is surprising that this terminology is not universals; these are not new ideas!

Note. The axioms of set theory give us ways to construct sets from existing sets (through unions, intersections, and other set operations). If we are not careful in imposing rules for the existence of sets then we can be led to contradictions, as we will discuss in the next section. We will argue that there is no "set of all sets," for example. We start with the Axiom of Separation which allows us to use a property to construct a "subset" of a set which we already know to exist.

Axiom of Separation. Given a set X and a property P, there is a set whose elements are the elements of X that have property P. That is, $\{x \mid x \in X \text{ and } P(x)\} = \{x \in X \mid P(x)\}$ is a set.

Example. If we accept that the set of integers exists, then we can use the Axiom of Separation to define the set $A = \{x \in \mathbb{Z} \mid x > 0\} = \mathbb{N}$. Here, the property P is that of being a positive integer. We can negate this consider the property $\sim P$ of being a non-positive integer. We then have the set $B = \{x \in \mathbb{Z} \mid x \leq 0\}$. Notice that we have then separated $\mathbb{Z}$ into sets A and B (thus the name of the axiom).

Note/Definition. By the Axiom of Separation, $\{x \in \mathbb{Z} \mid x \neq x\}$ is a set (assuming the set $\mathbb{Z}$ exists). Of course this set has no elements. It is the *empty set*, denoted $\varnothing$.

Note. We now state and prove (in a supplement to this section) a result about empty sets.

Theorem 2.5. There is exactly one empty set. That is, all empty sets are equal.

Example 2.6. We claim that $\varnothing \neq \{\varnothing\}$.

Proof. Recall that two sets are equal if they have the same members. The set $\varnothing$ has no members. The set $\{\varnothing\}$ has one member, namely $\varnothing$. So the sets are not equal and the claim holds. □