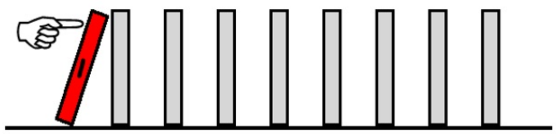
2.10. Mathematical Induction and Recursion

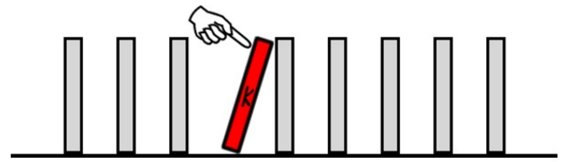
Note. In this section we give a technique that may be used to prove that a statement of the form $P(n)$ (where n is an integer greater than or equal to a given integer n_0) holds for all $n \geq n_0$. We give definitions and several examples. You may have encountered this idea in Calculus 1 (MATH 1910) when dealing with Riemann sums; you may recall the formula $\sum_{i=1}^{n} i = n(n+1)/2$, which can be demonstrated using this technique. For notes and examples in the Calculus 1 setting, see my online notes on Appendix A.2. Mathematical Induction and the corresponding video Appendix A.2 Video.

Note. Let "$P(n)$" denote the statement "The integer n has property P." Our goal is to prove that $P(n)$ is true for all integers $n \geq n_0$, where n_0 is some given integer. We use the technique of "mathematical induction" which allows us to prove an infinite number of statements in a finite number of steps! In Example 2.64, Gerstein gives some arguably artificial illustrations of the use of mathematical induction in the "real world."

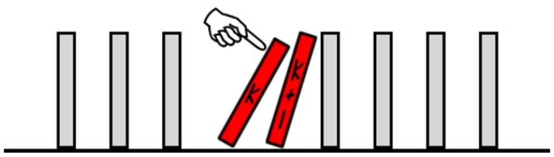
Note 2.10.A. Here is an informal way to think about induction which is clear and intuitive. Imagine an infinite line of dominos, where the dominos are labeled with consecutive integers greater than or equal to integer n_0, according to their position in the line. We want to describe a way to knock down all of the dominos. We do this by showing two things: (1) the first domino (the one labeled n_0) falls, and (2) when the domino labeled k falls, it knocks over the next domino in the line (the one labeled $k+1$). A nice collection of illustrations of this idea is the following (in which $n_0 = 1$), from Coolmath.com (accessed 1/7/2022):



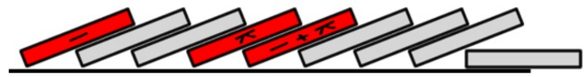
Check that the formula holds for $n = 1$.



Assume the formula holds for $n = k$.



Show that the assumption *implies* that the formula holds for $n = k + 1$.



Conclude that the formula holds for all $n \in \mathbb{N}$.

Note. Formally, the Principle of Mathematical Induction is based on a property the natural numbers. The *Well-Ordering Principle* for $\mathbb{N}$ states: "Every nonempty set of natural numbers has a least element." This property can be proved in an axiomatic development of the natural numbers. See my online notes for Introduction to Set Theory on Section 3.2. Properties of Natural Numbers (see Theorem 3.2.4). We state and prove the Principle of Mathematical Induction, based on the Well-Ordering Principle.

Theorem 2.66. The Principle of Mathematical Induction.
Let n_0 be an integer. Suppose P is a property such that

(a) $P(n_0)$ is true.

(b) For every integer $k \geq n_0$, the following conditional statement is true:

$$\text{If } P(n) \text{ is true for every } n \text{ satisfying } n_0 \leq n \leq k, \text{ then } P(k+1) \text{ is true.}$$

The $P(n)$ is true for every integer $n \geq n_0$.

Note. The statement of Theorem 2.66 is somewhat different that the falling dominos described in Note 2.10.A. Theorem 2.66 is sometimes called the "Strong Principle of Induction." This is contrast to the "Weak Principle of Induction" which replaces the conditional statement of (b) with the conditional statement:

If $P(n)$ is true for every $n = k$, then $P(k+1)$ is true.

Notice that the falling dominos story illustrates the Weak Principle. It can be shown that these are actually equivalent; see Proof of the Equivalence of Strong & Regular Induction on the Emory University Department of Mathematics and Computer Science webpage (accessed 1/7/2022). Quite often we will only require the Weak Principle.

Definition. In The Principle of Mathematical Induction, the act of verifying $P(n_0)$ is the *basis step* of the induction proof, and verifying statement (b) is the *induction step*. The hypothesis of the conditional statement in part (b) is the *induction hypothesis*.

Example 2.67. We claim that for every $n \in \mathbb{N}$,

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

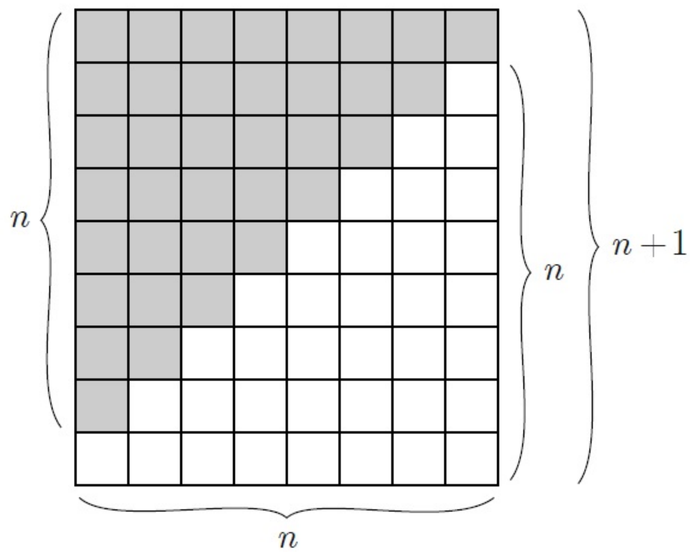
A plausible argument when n is even can be given by pairing the numbers $1, 2, \ldots, n-1, n$ as:

$$1 + 2 + \cdots + n/2 + (n/2+1) + \cdots + (n-1) + n$$

$$= (1+n) + (2+(n-1)) + (3+(n-2)) + \cdots + (n/2 + (n/2+1))$$

$$= (n/2)(n+1) = n(n+1)/2.$$

When n is odd, a similar argument can be given, but there is a "center" term of $(n+1)/2$ and the pairing procedure has to be modified. Geometrically, consider the following figure (from page 100 of the text book):



Notice that the area of the $n \times (n+1)$ rectangle is twice the are in white, which is $1+2+\cdots+n$. That is, $2(1+2+\cdots+n) = n(n+1)$, or $1+2+\cdots+n = n(n+1)/2$. These are nice motivational arguments, but they are not proofs. We can use the (Weak) Principle of Mathematical Induction to give a formal proof.

Note. We made a claim about the size of the power set of a finite set in Section 2.7. The Power Set. We now have the equipment to prove the claim.

Theorem 2.69. If S is a set with n elements then the power set $P(S)$ has 2^n elements.

Example 2.70. The example is a preview of the topic of congruence that we will encounter in Section 6.4. Congruence; Divisibility Tests. We claim that for every integer $n \geq 0$, the number $4^{2n+1} + 3^{n+2}$ is a multiple of 13. (Recall that integer x is a *multiple* of an integer y if $x = yt$ for some integer t.) We prove the claim using induction.

Note. Recall that a *prime number* is an integer $p > 1$ that has no integer factorization $p = ab$ in which both $a > 1$ and $b > 1$. The next theorem is a result from number theory which we can prove using the (Strong) Principle of Mathematical Induction.

Theorem 2.71. Every integer $n \geq 2$ is a product of primes numbers.

Note. Gerstein describes recursion as "an induction-like format for certain mathematical definitions." An example of a recursive definition of the sequence $s_n = 1 + 2 + 3 + \cdots + (n-1) + n$ is:

$$\begin{aligned} s_1 &= 1 \\ s_{k+1} &= (k+1) + s_k. \end{aligned}$$

The recursion is the fact that the "new term" s_{k+1} is defined as a function of the previous terms. The condition $s_1 = 1$ is the "initial condition." Another example of a recursively defined sequence is the *Fibonacci sequence* $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

where each new term is the sum of the previous two terms.

$$s_0 = 0, \quad s_1 = 1$$
$$s_{k+1} = s_{k-1} + s_k.$$

Notice that it is trickier to determine the nth term, s_n, in this sequence than it is in the example above. You may have seen the Fibonacci sequence in Linear Algebra (MATH 2010), in the setting of eigenvalues and diagonalizing a matrix. See my online Linear Algebra notes on Section 5.3. Two Applications. In the linear algebra setting, we can define the initial condition as the vector $\begin{bmatrix} s_1 \\ s_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and then we consider the matrix equation

$$\begin{bmatrix} s_{k+1} \\ s_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^k \vec{x}.$$

By diagonalizing matrix A we can find a (surprising) formula for the nth term of the Fibonacci sequence: $s_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)k\right)$. If this section we consider recursively defined sequences for which we can justify a formula for the nth term using induction.

Note. You are familiar with the summation symbol $\sum$ from Riemann sums in Calculus 1 (see my online notes for Calculus 1 [MATH 1910] on Section 5.2. Sigma Notation and Limits of Finite Sums): $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$. Similarly, there is the product symbol $\prod$ which represents the product of an indexed collection of numbers: $\prod_{i=1}^{n} a_i = a_1 \times a_2 \times \cdots \times a_{n-1} \times a_n$. We can use the product symbol to define the factorial symbol: $n! = \prod_{i=1}^{n} i = 1 \times 2 \times \cdots \times (n-1) \times n$. We can recursively define these symbols. The summation symbol is defined by

$$\begin{aligned} \sum_{i=1}^{1} a_1 &= a_1 \\ \sum_{i=1}^{k+1} a_i &= \left(\sum_{i=1}^{k} a_i \right) + a_{k+1}, \end{aligned}$$

the product symbol is defined by

$$\begin{aligned} \prod_{i=1}^{1} a_1 &= a_1 \\ \prod_{i=1}^{k+1} a_i &= \left(\prod_{i=1}^{k} a_i \right) \times a_{k+1}, \end{aligned}$$

and the factorial symbol is defined by

$$\begin{aligned} 0! &= 1 \\ (k+1)! &= k!(k+1). \end{aligned}$$

Revised: 1/8/2022